Inequalities

Jensen's Inequality: Assume that φ is convex on (a,b) and that the random variable X takes values in (a,b). Assume also that X, $\varphi(X)$ are integrable. Then

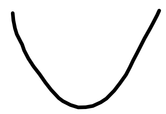
$$\varphi(E(X)) \leq E[\varphi(X)]$$

Lyapounov's inequalities: Assume that X is a r.v.

(i) If $p \geq 1$, then

$$E[|X|]^p \leq E[|X|^p]$$

$y = |x|^p$



(ii) If $1 \leq p \leq q$, then

$$E[|X|^p]^{1/p} \leq E[|X|^q]^{1/q}$$

$$\|X\|_p \leq \|X\|_q$$

Proof: (i) Apply Jensen's inequality to X and $\varphi(x) = |x|^p$.

$$E[|X|]^p = \varphi(E(|X|)) \leq E[\varphi(|X|)] \leq E[|X|^p]$$

(ii) Apply Jensen's inequality to $Y = |X|^p$ and $\underbrace{\varphi(x) = |x|^{q/p}}_{\text{convex since } q/p \geq 1}$.

$$E[|X|^p]^{q/p} = E[Y]^{q/p} = \varphi(E[Y]) \leq E[\varphi(Y)] = E[(\underbrace{|X|^p}_{Y})^{q/p}] = E[|X|^q]$$

Hence

$$E[|X|^p]^{q/p} \leq E[|X|^q]$$, take both sides to the power $1/q$

$$E[|X|^p]^{1/p} \leq E[|X|^q]^{1/q}$$

Modes of convergence

Question: What should it mean for a sequence $\{X_n\}$ of random variables to converge to a random variable X?

A lot of different things.

Pointwise convergence: $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega$.

Almost sure/everywhere convergence: There is a null set N such that $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \notin N$.

$P(N) = 0$.

Convergence in measure/probability: For $\varepsilon > 0$,

$$P\{\omega : |X(\omega) - X_n(\omega)| \geq \varepsilon\} \to 0 \quad \text{as } n \to \infty.$$

Convergence in mean/L^1: $\lim_{n\to\infty} E[|X - X_n|] = 0$.

Convergence in L^p ($p \geq 1$): $\lim_{n\to\infty} E[|X - X_n|^p] = 0$.

Convergence in distribution: Let F_n and F be the distribution functions of X_n and X.

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{at all points where } F \text{ is continuous.}$$

Advantage: The random variable can be defined on different prob. space.

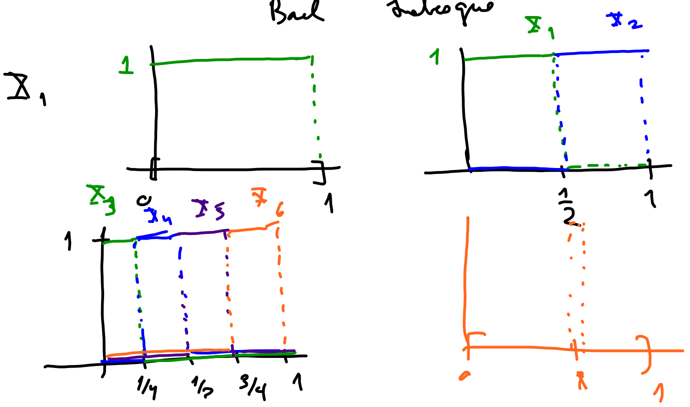
Example: A sequence converging in prob and L^1, but not converging pointwise.

$X_n \to 0$ in prob and L^1

$X_n \not\to 0$ pointwise

$(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, P)$

Borel, Lebesgue



Proposition: If $X_n \to X$ in L^p, then $X_n \to X$ in probability.

Proof: Chebyshev's inequality

$$P\{\omega : |X - X_n| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} E\left[|X - X_n|^p\right] \to 0,$$

hence convergence in probability.

Proposition: If $X_n \to X$ almost surely, then $X_n \to X$ in probability.

Proof: Note that there is a set Λ of prob. 0 such that $\lim_{n\to\infty} X_n(\omega) = X(\omega)$. Let $Z_n(\omega) = \sup\{|X_k(\omega) - X(\omega)| : k \geq n\}$

If $\omega \notin \Lambda$, then $\lim_{n\to\infty} Z_n(\omega) = 0$. Hence given an $\varepsilon > 0$, let $T_n^\varepsilon := \{\omega : Z_n(\omega) \geq \varepsilon\}$. Then

$$\bigcap_{n\in\mathbb{N}} T_n^\varepsilon \subseteq \Lambda.$$

Hence

$$0 = P\Big(\bigcap_{n\in\mathbb{N}} T_n^\varepsilon\Big) \overset{\text{cont meas}}{=} \lim_{n\to\infty} P(T_n^\varepsilon)$$

But then

$$P\{\omega \mid |X_n(\omega) - X(\omega)| \geq \varepsilon\} \overset{?}{\leq} P(T_n^\varepsilon) \to 0$$

Hence $X_n \to X$ in probability.

Problems

2.37, 3.2, 3.3, 3.4, 3.5, 3.6
3 1 2.

3.3: Show that if $X \geq 0$ a.s., then $E[X] = 0$ iff $X = 0$ a.s.

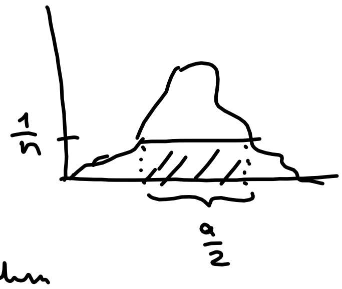
Assume $X = 0$ a.s. Then $E[X] = E[0] = 0$.

Assume X is not equal to 0 a.s. Then $P\{\omega : X(\omega) > 0\} = a > 0$. Note

$$\{\omega : X(\omega) > 0\} = \bigcup_{n \in \mathbb{N}} \{\omega : X(\omega) > \tfrac{1}{n}\}$$ increasing

$a = P\{\omega : X(\omega) > 0\} = \lim_{n \to \infty} P\{\omega : X(\omega) > \frac{1}{n}\}$, so for large enough n, $P\{\omega : X(\omega) > \frac{1}{n}\} > \frac{a}{2}$.

$$E[X] \geq \frac{a}{2} \cdot \frac{1}{n}.$$



3.5 Show that if X is integrable, then

$$\lim_{x \to \infty} x\, P[|X| \geq x] = 0.$$

By replacing X by $|X|$ if necessary, we may assume that $X \geq 0$.

$$x P[X \geq x] \leq x P[\bar{X}_0 \geq x] = x \sum_{k \geq x} P[\bar{X}_0 = k]$$

$$= \sum_{k \geq x} x P[\bar{X}_0 = k] \leq \underbrace{\sum_{k \geq x} k P[\bar{X}_0 = k]}_{\text{tale of a convergent sequence}}$$

$$E[\bar{X}_0] = \sum_{k=0}^{\infty} k P[\bar{X}_0 = k] < \infty$$ since $\bar{X}_0$ integrable

Hence $\lim_{x \to \infty} \sum_{k \geq x} k P[\bar{X}_0 = k] = 0$.

Abel summation: Assume $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ are two sequences.

Define
$$s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$$
and assume that $\lim_{n\to\infty} s_n b_n = 0$. Then
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=0}^{\infty} s_k (b_k - b_{k+1})$$
in the sense that if one series converges, so does the other and the two sums are equal.

Proof: Note that $a_k = s_k - s_{k-1}$ (holds even for $k=1$ if we take $s_0 = 0$)

$$\sum_{k=1}^{N} a_k b_k = \sum_{k=1}^{N} (s_k - s_{k-1}) b_k = \sum_{k=1}^{N} s_k b_k - \sum_{k=2}^{N} s_{k-1} b_k$$
$$= \sum_{k=1}^{N} s_k b_k - \sum_{k=1}^{N-1} s_k b_{k+1} = s_N b_N + \sum_{k=1}^{N-1} (s_k b_k - s_k b_{k+1})$$
$$= \underbrace{s_N b_N}_{\downarrow 0} + \sum_{k=1}^{N-1} s_k (b_k - b_{k+1})$$

let $N \to \infty$, then
$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} s_k (b_k - b_{k+1})$$

Problem 2.37 X is nonnegative and takes integer values $0, 1, 2, 3, \dots$

Prove that $E[X] = \sum_{n=1}^{\infty} P[X \geq n]$.

Know that: $E[X] = \sum_{n=1}^{\infty} n P[X = n]$

$$P[X = n] = \underbrace{P[X \geq n]}_{b_n} - \underbrace{P[X \geq n+1]}_{b_{n+1}}$$

Abel summation:
$$\sum_{n=1}^{\infty} \overbrace{1}^{a_n} \cdot \underbrace{P[X \geq n]}_{b_n} = \sum_{n=1}^{\infty} \overset{n}{\overset{\shortparallel}{s_n}} \underbrace{\left[P[X \geq n] - P[X \geq n+1]\right]}_{P(X = n)}$$
$$= \sum_{n=1}^{\infty} n P[X = n] = \sum_{n=0}^{\infty} n P[X = n] = E[X]$$

Problem solved provided that the condition for Abel summation is satisfied: $\lim_{n\to\infty} s_n b_n = 0$ i.e.

$\lim_{n\to\infty} n P[X \geq n] = 0$ by problem 3.5.

Sketch of 3.4: $X \geq 0$ r.v. F distribution function

$$E[X] = \int_0^\infty (1-F(x))\,dx$$

$$E[X] \leftarrow E[\bar{X}_n] = \sum_{k=0}^\infty \frac{k}{2^n} P\left(\frac{k-1}{2^n} < X \le \frac{k}{2^n}\right)$$

$$= \frac{1}{2^n}\sum_{k=0}^\infty \underbrace{k}_{s_k} \left\{ \underbrace{P\left[X > \frac{k-1}{2^n}\right]}_{b_k} - \underbrace{P\left[X > \frac{k}{2^n}\right]}_{b_{k+1}} \right\} \qquad a_k = 1$$

$$= \frac{1}{2^n}\sum_{k=0}^\infty \overset{a_k}{1} \cdot \underbrace{P\left[X > \frac{k-1}{2^n}\right]}_{b_k}$$

$$= \frac{1}{2^n}\sum_{k=1}^\infty \left(1 - P\left[X \le \frac{k-1}{2^n}\right]\right) = \underbrace{\sum_{k=1}^\infty \left(1 - F\left(\frac{k-1}{2^n}\right)\right)\frac{1}{2^n}}_{\text{Riemann su}}$$

$$\to \int_0^\infty (1-F(u))\,du$$