Review

$(\Omega, \mathcal{F}, P)$

Family of sets: a collection $\mathcal{F}$ of subsets of Ω is a σ-algebra if

(i) $\emptyset \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) If $A_n \in \mathcal{F}$ for each $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$

If $\mathcal{G}$ is any collection of subsets of Ω, $\sigma(\mathcal{G})$ is the σ-algebra generated by $\mathcal{G}$, i.e. the smallest σ-algebra containing $\mathcal{G}$

Example: $\mathcal{B}$ - the Borel σ-algebra on $\mathbb{R}$

$\mathcal{B} = \sigma(\mathcal{G})$ — $\mathcal{G}$ is the family of all open subsets of $\mathbb{R}$
— $\mathcal{G}$ is the family of — open intervals (a,b) — closed intervals $[a,b]$ — half-open intervals $(a,b]$

If X is a random variable

$$\sigma(X) = \sigma\{\underbrace{[X \in B]}_{X^{-1}(B)} : B \text{ is a Borel set}\}$$

Strategy: Assume that we want to prove that a certain property P holds for all sets in $\sigma(\mathcal{G})$. Then

(i) Show that all elements in $\mathcal{G}$ have property P

(ii) Show that the collection of all sets with property P forms a σ-algebra
 (i) $\emptyset$ has prop P
 (ii) A has P $\Rightarrow A^c$ has P
 (iii) A_n has P for all n, then $\bigcup A_n$ has P

Then all elements in $\sigma(\mathcal{G})$ has property P.

Algebra: $\mathcal{A}$ a collection of subsets of Ω. $\mathcal{A}$ is an algebra if

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$ ($\Rightarrow A_1 \cup \dots \cup A_n \in \mathcal{F}$)

Monotone Class: a collection $\mathcal{M}$ of subsets of Ω is a monotone class if

Same with cont of measure.
(i) If $\{A_n\}$ is an increasing sequence of sets in $\mathcal{M}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$.
(ii) -"- a decreasing -"- , then $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{M}$.

Monotone Class Theorem: If $\mathcal{A}$ is an algebra, then $\sigma(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

Second strategy for proving that all sets in $\sigma(\mathcal{A})$ has a property P (when $\mathcal{A}$ is an algebra!)

(i) Check that sets in $\mathcal{A}$ has P

(ii) Check that the sets with property P is a monotone class.

Then all sets in $\sigma(\mathcal{A})$ has property P.

σ-algebras as information

$\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{G}$ codes information.

Filtration: Timeline $\mathbb{N}_0$

$\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s < t$

$\mathcal{F}_t \subseteq \mathcal{F}$ for all t.

Idea: $\mathcal{F}_t$ represents the information available at time t.

If $A \in \mathcal{F}_t$, you should at time t be able to tell whether an element ω is in A or not.

Conditional expectations: $X : \Omega \to \mathbb{R}$ integrable random variable

A random variable Y is the conditional expectation $E[X|\mathcal{G}]$ if

(i) Y is $\mathcal{G}$-measurable

(ii) $\int_\Lambda Y \, dP = \int_\Lambda X \, dP$ for all $\Lambda \in \mathcal{G}$

Intuition: $E[X|\mathcal{F}_t]$ the best prediction of X from what is known at time t.

Properties: $E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}]$ a.s

$X \leq Y \Rightarrow E[X|\mathcal{G}] \leq E[Y|\mathcal{G}]$ a.s

$E[XY|\mathcal{G}] = Y E[X|\mathcal{G}]$ if Y is $\mathcal{G}$-measurable.

$E[X|\mathcal{G}] = E[X]$ for X independent of $\mathcal{G}$

$\mathcal{G}_1 \subseteq \mathcal{G}_2$ $\quad E[X|\mathcal{G}_1] = E[\underline{E[X|\mathcal{G}_2]}|\mathcal{G}_1]$

Martingales: $\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$ a filtration: Let $\{X_t\}_{t \in \mathbb{N}_0}$ be a stochastic process. Then

(i) Each X_t is integrable

(ii) X is adapted (i.e. X_t is $\mathcal{F}_t$-measurable)

(iii) If for all $s < t$
$$\begin{cases} E[X_t|\mathcal{F}_s] \geq X_s, \text{ then } X \text{ is a } \underline{\text{submartingale}} \\ E[X_t|\mathcal{F}_s] \leq X_s, \text{ — '' — } \underline{\text{supermartingale}} \\ E[X_t|\mathcal{F}_s] = X_s \text{ — '' — } \underline{\text{martingale}} \end{cases}$$

Doob's Decomposition Theorem: A submartingale is the sum of a martingale plus an adapted increasing process.

Stopping times: $T : \Omega \to \mathbb{N}_0$ is a stopping time if

$[T = n] \in \mathcal{F}_n$ for all n.

equivalently

$[T \leq n] \in \mathcal{F}_n$ — '' —

Extended filtration: If T is a stopping time, the σ-algebra $\mathcal{F}_T$ is given by

$\Lambda \in \mathcal{F}_T \Leftrightarrow \Lambda \cap [T \leq n] \in \mathcal{F}_n$ for all n

$\overset{\text{eq}}{\Leftrightarrow} \Lambda \cap [T = n] \in \mathcal{F}_n$ — '' —

Optional Sampling: If X is a submartingale and $S \leq T$ are finite stopping times, then if either

(i) S, T are bounded

or (ii) X is bounded

then $E[X_T|\mathcal{F}_S] \geq X_S$.

Probability measures

$(\Omega, \mathcal{F}, P)$ where

(i) $P(\emptyset) = 0$ and $P(\Omega) = 1$ ←

(ii) If $\{A_n\}$ is a disjoint sequence of sets in $\mathcal{F}$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Continuity of measure:

(i) $A_n \nearrow A$, then $P(A) = \lim_{n\to\infty} P(A_n)$ [here $A = \cup A_n$]

(ii) $A_n \searrow A$, then $P(A) = \lim_{n\to\infty} P(A_n)$ [here $A = \cap A_n$]

Examples: Lebesgue measure: $\mu((a,b]) = b - a$.

X random variable induces a prob. measure on $\mathbb{R}$

$$\mu(B) = P[X \in B] = P[X^{-1}(B)] \leftarrow \text{distribution of } X$$

Independence: A, B are independent if

$$P(A \cap B) = P(A)P(B) \rightarrow P(A|B) = \frac{P(A\cap B)}{P(B)}$$

The sets $\{A_i\}_{i \in I}$ is independent if whenever we choose distinct indices $i_1, i_2, \ldots, i_n$, then

$$P(A_{i_1} \cap \ldots \cap A_{i_n}) = P(A_{i_1}) \ldots P(A_{i_n}).$$

Random variables

$X : \Omega \to \mathbb{R}$ is a **random variable** if for all $x \in \mathbb{R}$ we have

$\{\omega : X(\omega) \le x\} \in \mathcal{F}$

or equivalently

$\{\omega : X(\omega) < x\} \in \mathcal{F}$

or $\{\omega : X(\omega) \ge x\} \in \mathcal{F}$

or $\{\omega : X(\omega) > x\} \in \mathcal{F}$.

Expectations:

Step 1: The discrete case: $X = \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}$ ($\{A_i\}$ a partition of Ω)

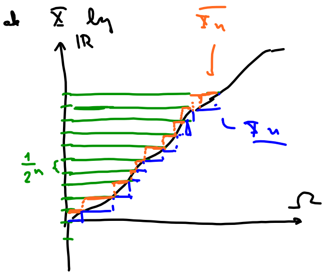
X is **integrable** if $\sum_{i=1}^{\infty} |a_i| P(A_i) < \infty$ and

$$E[X] = \sum_{i=1}^{\infty} a_i P(A_i)$$

Step 2: The general case: Approximate X by

$$\overline{X}_n = \sum_{k=-\infty}^{\infty} \frac{k+1}{2^n} \mathbb{1}_{\left(\frac{k}{2^n} < X \le \frac{k+1}{2^n}\right]}$$

$$\underline{X}_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbb{1}_{\left(\frac{k}{2^n} < X \le \frac{k+1}{2^n}\right]}$$



$\overline{X}_n \downarrow X$, $\underline{X}_n \uparrow X$

Def: X is **integrable** if $\overline{X}_0$ is integrable and

$$E[X] = \lim_{n\to\infty} E[\overline{X}_n] = \lim_{n\to\infty} E[\underline{X}_n]$$

Notation: $E[X]$ is also written $\int X \, dP$

$$\int_\Lambda X \, dP = \int \mathbb{1}_\Lambda X \, dP = E[\mathbb{1}_\Lambda X]$$

Connections to independence: X and Y are independent if

(i) $P[X \le \alpha \wedge Y \le \beta] = P[X \le \alpha] P[Y \le \beta]$

or equivalently

(ii) $P[X \in A \wedge Y \in B] = P[X \in A] P[Y \in B]$ (Borel sets A, B)

The family $\{X_i\}_{i \in I}$ is **independent** if for all distinct $i_1, i_2, \dots, i_n$ we have

$$P[X_{i_1} \le \alpha_1 \wedge X_{i_2} \le \alpha_2 \wedge \dots \wedge X_{i_n} \le \alpha_n]$$
$$P[X_{i_1} \le \alpha_1] \cdot \ldots \cdot P[X_{i_n} \le \alpha_n]$$

If X, Y are independent $E[XY] = E[X] E[Y]$

$X_1, \dots, X_n$ — " — $E[X_1 X_2 \dots X_n] = E[X_1] \dots E[X_n]$

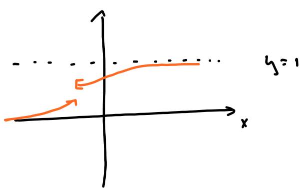
Distribution: X r.v.

(i) **Distribution function:** $F_X(x) = P[X \le x]$

Properties: (i) F_X is increasing.

(ii) F_X is right continuous

(iii) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$



(ii) **Distribution of X:** A Borel measure μ_X on $\mathbb{R}$ s.t.

$$\mu_X(B) = P[X \in B]$$

Note that $\mu_X((-\infty, x]) = P[X \le x] = F_X(x)$

Relationship:

Notation.

$$E[g(X)] = \int g(x) \, d\mu_X(x) = \int g(x) \, dF_X(x)$$

(iii) **Density:** $\int g(x) \, dF_X(x) = \int_{-\infty}^{\infty} g(x) f(x) \, dx$