Review

Modes of convergence

What can it mean $X_n \to X$?

Convergence almost surely: There is a set $\Omega' \in \Omega$ with $P(\Omega') = 1$ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega'$.

Convergence in probability: For all $\varepsilon > 0$

$$\lim_{n\to\infty} P[\,|X - X_n| \geq \varepsilon\,] = 0.$$

Convergence in expectation (or in L^1): $E[\,|X - X_n|\,] \to 0$ as $n \to \infty$

Convergence in L^p for $p \geq 1$: $E[\,|X - X_n|^p\,] \to 0$ as $n \to \infty$

Convergence in distribution: $F_{X_n}(x) \to F_X(x)$ at all points x where F_X is continuous.

or equivalently

$$E[f(X_n)] \to E[f(X)] \quad \text{as } n \to \infty$$

for all bounded, continuous f.

Relationships:

convergence a.s. $\Rightarrow$ convergence in probability
$\Rightarrow$ convergence in distribution

convergence in $\underbrace{\text{expectation}}_{L^p}$ $\Rightarrow$ convergence in prob

convergence in prob $\Rightarrow$ convergence in distribution
$\Rightarrow$ there is a subsequence $\{X_{n_k}\}$ that converges to X a.s.

Analytic Limit Theorems

Basic question: When is $\lim_{n\to\infty} E[X_n] = E[\lim_{n\to\infty} X_n]$?

Monotone Convergence Theorem: Assume that $\{X_n\}$ is an increasing sequence of positive random variables. Then

$$\lim_{n\to\infty} E[X_n] = E[\lim_{n\to\infty} X_n]$$

Fatou's Lemma: Assume that $\{X_n\}$ is a sequence of positive random variables. Then

$$\liminf E[X_n] \geq E[\liminf X_n]$$

Dominated Convergence Theorem: Assume that $\{X_n\}$ is a sequence of random variables converging a.s or in probability to X. Assume that there is an integrable random variable Y such that $|X_n| \leq Y$.

Then $$\lim_{n\to\infty} E[X_n] = E[X] = E[\lim_{n\to\infty} X_n]$$

Probabilistic Limit Theorems

Laws of Large Numbers: Conditions $\Rightarrow$ $\dfrac{X_1+X_2+\cdots+X_n}{n} \to 0$

Central Limit Theorem: Conditions $\Rightarrow$ $\dfrac{X_1+X_2+\cdots+X_n}{\sqrt{n}} \to$ normal distribution

Weak Law of Large Numbers: If $\{X_n\}$ is an independent sequence of random variable with mean 0 and $E[X_n^2] \leq \sigma^2$, then

$$\frac{X_1+X_2+\cdots+X_n}{n} \to 0 \text{ in probability}$$

Strong Law of Large Numbers (Cantelli). If $\{X_n\}$ is an independent sequence of random variable with mean 0 and $E[X_n^4] \leq M$. Then

$$\frac{X_1+X_2+\cdots+X_n}{n} \to 0 \text{ almost surely}$$

Central Limit Theorem (iid case). Assume that $\{X_n\}$ is a iid-sequence of random variables with mean μ and variance $\sigma^2 > 0$. Then

$$\frac{X_1+X_2+\cdots+X_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N(0,1) \text{ in prob.}$$

Central Limit Theorem (Lyapunov's version) Assume that $\{X_n\}$ is an independent sequence of random variables with mean zero and finite variances σ_n^2. Let $S_n = X_1+X_2+\cdots+X_n$ and let $s_n^2 = \sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2$. Assume in addition that $\gamma_j = E[|X_j|^3]$ exists and that $\dfrac{\sum_{j=1}^{n}\gamma_j}{s_n^3} \to 0$ as $n \to \infty$. Then

$$\frac{S_n}{s_n} \Rightarrow N(0,1) \text{ in distribution.}$$

Lévy Continuity Theorem: If $\{X_n\}$ is a sequence of random variables with characteristic functions φ_n and $\varphi_n(t) \to \varphi(t)$ for all t to a function φ continuous at 0, then $\{X_n\}$ converges in distribution to a random variable X, whose characteristic function is φ.

0-1 Laws

$\{A_n\}$ a sequence of events:

$\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \{\omega : \omega$ belongs to infinitely many $A_n\}$

$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \{\omega : \omega$ belongs to all but finitely many $A_n\}$

Borel-Cantelli:

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P[\limsup A_n] = 0$.

(ii) Assume that $\{A_n\}$ is independent. Then

a) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P[\limsup A_n] = 0$.

b) If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P[\limsup A_n] = 1$

Assume that $\{X_n\}$ is a sequence of random variables and put

$\mathcal{F}_n^* = \sigma(X_{n+1}, X_{n+2}, \dots)$. Then $\mathcal{F}_\infty^* = \bigcap_{n=1}^{\infty} \mathcal{F}_n^*$ is a σ-algebra.

Borel/Kolmogorov 0-1-Law: If $\{X_n\}$ is an independent sequence of random variables and $\Delta \in \mathcal{F}_\infty^*$, then $P(\Delta)$ is either 0 or 1.

Characteristic functions

X random variable:

$$\varphi_X(t) = E[e^{itX}] = E[\cos tX] + i\,E[\sin tX]$$

$$\| \int e^{itx}\, d\mu_X(x) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

if a density exist.

Thm: If X has n-th moment, then φ_X is n times differentiable and

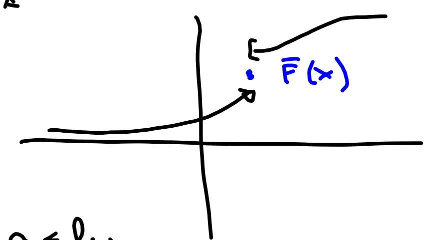
$$\varphi_X(t) = \sum_{j=1}^{n} \frac{(it)^j}{j!} E[X^j] + o(t^n)$$

Independence: If $X_1, X_2, \ldots, X_n$ are independent

$$\varphi_{X_1+X_2+\cdots+X_n}(t) = \varphi_{X_1}(t)\,\varphi_{X_2}(t)\cdots\varphi_{X_n}(t)$$

X is normal $N(\mu,\sigma^2)$, then $\varphi_X(t) = e^{-\frac{\sigma^2 t^2}{2} + i\mu t}$

$$\bar F(x) = \frac{F(x)+F(x-)}{2}$$



Lévy's inversion formula: For any $a<b$,

$$\bar F(b) - \bar F(a) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{e^{-itb} - e^{-ita}}{-2\pi i t}\, e^{-\frac{\varepsilon^2 t^2}{2}}\, \varphi(t)\, dt.$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-2\pi i t}\, \varphi(t)\, dt.$$

Corollary: If you know φ, you know F.

Inequalities

Chebyshev's inequality: $P[|X| \geq \lambda] \leq \frac{1}{\lambda^p} E[|X|^p]$

Schwarz' inequality: $E[|XY|] \leq E[X^2]^{1/2} E[Y^2]^{1/2}$

$$|\langle x,y\rangle| \leq \|x\| \, \|y\|$$

Lyapounov's inequalities: $1 \leq p \leq q$:

(i) $E[|X|]^q \leq E[|X|^q]$

(ii) $\underbrace{E[|X|^p]^{1/p}}_{\|X\|_p \leq \|X\|_q} \leq E[|X|^q]^{1/q}$

Jensen's inequalities: Convex φ & ......

(i) $\varphi(E[X]) \leq E[\varphi(X)]$

(ii) $\varphi(E[X|\mathcal{G}]) \leq E[\varphi(X)|\mathcal{G}]$ a.s.

Martingal maximal inequality: X a positive submartingale:

$$P[\underbrace{\max_{k \leq n} X_k}_{X_n^*} \geq \lambda] \leq \frac{1}{\lambda} E[|X_n|]$$