Convergence in distribution

Last time:

Theorem: Assume that $\{X_n\}$ is a sequence of r.v. Then $\{X_n\}$ converges in distribution to X if and only

$$E[f(X_n)] \to E[f(X)]$$

for all bounded, continuous $f : \mathbb{R} \to \mathbb{R}$.

Notation: $\{X_n\}, \{F_n\}, \{\mu_n\}$

$$\left.\begin{array}{l} X_n \Rightarrow X \\ F_n \Rightarrow F \\ \mu_n \Rightarrow \mu \end{array}\right\} \text{ convergence in distribution.}$$

Helly's Theorem: Let $\{F_n\}$ be a sequence of distribution functions. Then there is a subsequence $\{F_{n_k}\}$ converging pointwise to a limit function F_∞. Assume that $\lim_{x \to -\infty} F_\infty(x) = 0$ and $\lim_{x \to \infty} F_\infty(x) = 1$. Then there is a distribution function F such that $F_{n_k} \Rightarrow F$.

Proof: Recall the Bolzano-Weierstrass Theorem (MAT 1110):
Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Let $r_1, r_2, r_3, \ldots$ be an enumeration of $\mathbb{Q}$.

Then by BW, the seq. $\{F_n(r_1)\}$ now has a convergent subsequence. Let

$$F_1^\alpha, F_2^\alpha, \ldots\ldots$$

be the functions occurring in that subsequence. We know $F_1^\alpha(r_1), F_2^\alpha(r_1), \ldots$ converges.

Now look at $F_1^\alpha(r_2), F_2^\alpha(r_2), F_3^\alpha(r_2), \ldots\ldots$ By BW, there is a convergent subsequence, and we let

$$F_1^\beta, F_2^\beta, F_3^\beta$$

be the functions occurring in that subsequence. Then $\{F_n^\beta(r_1)\}, \{F_n^\beta(r_2)\}$ both converge.

Next look at $\{F_n^\beta(r_3)\}$. Again there is a convergent subsequence, and if we pick the the functions from that subsequence, we get a new subsequence $\{F_n^\gamma\}$. Note $\{F_n^\gamma(r_1)\}, \{F_n^\gamma(r_2)\}, \{F_n^\gamma(r_3)\}$ converge.

Now make a new subsequence by picking

$$F_1^\alpha, F_2^\beta, F_3^\gamma, \ldots\ldots$$

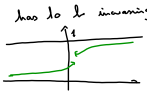
If r_i is a rational number, the sequence

$$F_1^\alpha(r_i), F_2^\beta(r_i), F_3^\gamma(r_i), \ldots\ldots$$

must converge.

Define $F_\infty : \mathbb{Q} \to \mathbb{R}$ by letting $F_\infty(r_i)$ be the limit.

Since F_∞ is the limit of distribution function it has to be increasing and take values between 0 and 1.



There can only be countably many points $x \in \mathbb{R}$ such that

$$\lim_{r \uparrow x} F_\infty(r) < \lim_{r \downarrow x} F_\infty(r)$$

If x is not of this kind, the sequence

$$F_1^\alpha(x), F_2^\beta(x), F_3^\gamma(x), \ldots$$

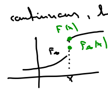
has to converge by squeezing.

Repeat the procedure from the beginning of the proof applied to the countably many jump points instead of the rationals. This will give us a new subsequence that converges even at the jump points. Call this subsequence $\{F_{n_k}\}$. Then $F_{n_k}(x) \to F_\infty(x)$ for all x.

Assume now that $\lim_{x \to -\infty} F_\infty(x) = 0$ and $\lim_{x \to \infty} F_\infty(x) = 1$. F_∞ need not be right continuous, but

$$F(x) = \lim_{y \downarrow x} F_\infty(y)$$

will be.

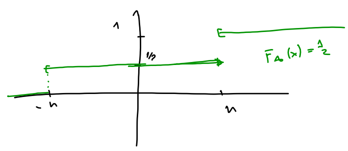


F is also increasing and we have $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$ since these properties hold for F_∞. Hence F is a distribution function. It still remain to prove that $F_{n_k} \Rightarrow F$.

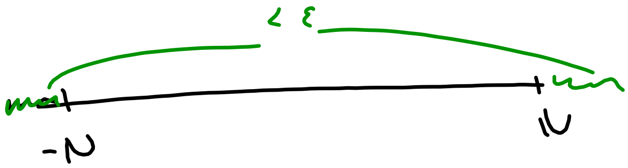
Let x be a cont. point of F. Then $F(x) = F_\infty(x)$. and $F_{n_k}(x) \to F_\infty(x) = F(x)$. Thus $F_{n_k} \Rightarrow F$.

Example: A situation where we do not get convergence in distribution

$$X_n = \begin{cases} n & \text{with prob } 1/2 \\ -n & \text{with prob } 1/2 \end{cases}$$



Definition: A sequence $\{\mu_n\}$ of distributions is called **tight** if for every $\varepsilon > 0$, there is a number N such that $\mu_n([-N,N]) < 1-\varepsilon$.



Corollary: Assume that $\{X_n\}$ is a tight sequence of random variables. Then there is a subsequence $\{X_{n_k}\}$ that converges in distribution.

Proof: Let F_n be the distribution functions. By Helly's Theorem there is a subsequence F_{n_k} converging pointwise to a function F_∞. The corollary will follow from Helly's Theorem if we can prove that $\lim_{x \to -\infty} F_\infty(x) = 0$, $\lim_{x \to \infty} F(x) = 1$.

Given an $\varepsilon > 0$, there is by tightness an N such that $\mu_n[-N,N] < 1-\varepsilon$, and hence $\mu_n[N,\infty]] < \varepsilon$.

Thus $F_n(N) > 1-\varepsilon$ for all n, and thus $F_\infty(N) > 1-\varepsilon$.

Hence $\lim_{x \to \infty} F_\infty(x) = 1$. Same argument for $-\infty$.

Weak convergence

Assume that $\mathbb{X}$ is a metric space, and let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{X}$. Let $\{\mu_n\}$ be a sequence of Borel measures. If μ is another Borel measure, we say that $\{\mu_n\}$ converges weakly to μ if

$$E_{\mu_n}[f] \to E_\mu[f]$$

for all bounded continuous functions.

Definition: We say that $\{\mu_n\}$ is tight if for every $\varepsilon > 0$, there is a compact set $K_\varepsilon \subseteq \mathbb{X}$ such that $\mu_n(K_\varepsilon) > 1-\varepsilon$ for all $n \in \mathbb{N}$.

The Helly-Bray Theorem: If $\{\mu_n\}$ is a sequence of prob. measures that is tight, then there is a prob. measure μ on $\mathbb{X}$ such that $\{\mu_n\}$ converges weakly to μ.

Lévy's continuity theorem

Prop: Assume $\mu_n \Rightarrow \mu$. If φ_n and φ are the characteristic functions, then $\varphi_n(t) \to \varphi(t)$ for all t.

Proof: $\varphi_n(t) = E[e^{it\mathbb{X}_n}] = E[\underbrace{\cos t\mathbb{X}_n}_{\cos(t\cdot x)}] + iE[\underbrace{\sin(t\mathbb{X}_n)}_{\sin(tx)}]$ bounded cont fund.

$$\xrightarrow{\text{conv. dist.}} E[\cos t\mathbb{X}] + iE[\sin t\mathbb{X}]$$

$$= E[e^{it\mathbb{X}}] = \varphi(t).$$

Lemma: Assume that μ is a distribution with characteristic function φ. Then for all $A > 0$, then

$$\underbrace{\mu([-2A, 2A])}_{\approx 1} \geq \underbrace{A\left|\int_{-1/A}^{1/A} \varphi(t)\,dt\right| - 1}_{\approx A\cdot\frac{2}{A} - 1 = 2-1=1}$$

Proof: Note for $T > 0$

$$\frac{1}{2T}\int_{-T}^{T} \varphi(t)\,dt = \int_{-\infty}^{\infty} \frac{\sin(Tx)}{Tx}\,d\mu(x)$$

because

$$\frac{1}{2T}\int_{-T}^{T}\varphi(t)\,dt = \frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} e^{itx}\,d\mu(x)\right]dt$$

$$= \frac{1}{2T}\int_{-\infty}^{\infty}\left[\int_{-T}^{T} e^{itx}\,dt\right]d\mu(x)$$

$$= \frac{1}{2T}\int_{-\infty}^{\infty}\left[\frac{e^{itx}}{ix}\right]_{t=-T}^{t=T} d\mu(x)$$

$$= \int_{-\infty}^{\infty} \frac{e^{iTx} - e^{-iTx}}{2Tix}\,d\mu(x) = \int_{-\infty}^{\infty}\frac{\sin Tx}{Tx}\,d\mu(x)$$

But then

$$\left|\frac{1}{2T}\int_{-T}^{T}\varphi(t)\,dt\right| = \left|\int_{-\infty}^{\infty}\frac{\sin Tx}{Tx}\,d\mu\right|$$

$$= \left|\int_{-2A}^{2A}\underbrace{\frac{\sin Tx}{Tx}}_{\leq 1}d\mu(x) + \int_{-\infty}^{-2A}\underbrace{\frac{\sin Tx}{Tx}}_{\leq \frac{1}{2TA}}d\mu(x) + \int_{2A}^{\infty}\frac{\sin Tx}{Tx}\,d\mu\right|$$

$$\leq \mu([-2A, 2A]) + \frac{1}{2TA}\mu((-\infty, 2A)) + \frac{1}{2TA}\mu([2A, \infty))$$

$$= \mu([-2A, 2A]) + \frac{1}{2\underbrace{TA}_{=1}}\left(1 - \mu[-2A, 2A]\right)$$

Let us choose $T = \frac{1}{A}$:

$$\left|\frac{A}{2}\int_{-1/A}^{1/A}\varphi(t)\,dt\right| \leq \mu([-2A, 2A]) + \frac{1}{2} - \frac{1}{2}\mu[-2A, 2A]$$

$$= \frac{1}{2}\mu([-2A, 2A]) + \frac{1}{2}$$

Hence

$$\mu([-2A, 2A]) \geq A\left|\int_{-1/A}^{1/A}\varphi(t)\,dt\right| - 1 \quad . \text{ Q.E.D.}$$

$\varphi_n(t) \to \varphi(t)$, ~~$\varphi(0)=1$~~ φ cont at 0.

$\mu_n \Rightarrow \mu$.