Lévy's Continuity Theorem

Last time: For all $A > 0$,
$$\mu([-2A, 2A]) \geq A \left| \int_{-1/A}^{1/A} \varphi(t)\, dt \right| - 1$$

Lévy's Continuity Theorem: Let $\{\mu_n\}$ be a sequence of distributions with characteristic functions φ_n. Assume that there is a function φ_∞ such that

(i) $\varphi_n(t) \to \varphi_\infty(t)$ for all

(ii) φ_∞ is continuous at 0.

Then

a) There is a distribution μ_∞ such that $\mu_n \Rightarrow \mu_\infty$.

b) φ_∞ is the characteristic function of μ_∞.

Proof: Plan: Use Helly's Theorem to prove that there is a convergent subsequence $\{\mu_{n_k}\}$ and that the limit μ_∞ has char. fund. φ_∞. Extend to the entire sequence.

Assume that $\varepsilon > 0$ is given. Since φ_∞ is continuous at the origin, there is a $\delta > 0$ such that $|\underbrace{1}_{\varphi_\infty(0)} - \varphi_\infty(t)| < \frac{\varepsilon}{4}$ when $|t| < \delta$. Now apply the lemma with $A = \frac{1}{\delta}$:

$$\mu_n([-\tfrac{2}{\delta}, \tfrac{2}{\delta}]) \geq \frac{1}{\delta} \left| \int_{-\delta}^{\delta} \varphi_n(t)\, dt \right| - 1$$
$$\geq \frac{1}{\delta} \left| \int_{-\delta}^{\delta} [\varphi_\infty(t) - (\varphi_\infty(t) - \varphi_n(t))]\, dt \right| - 1$$
$$\geq 2 \underbrace{\frac{1}{2\delta} \left| \int_{-\delta}^{\delta} \varphi_\infty(t)\, dt \right|}_{\text{mean value}} - \frac{1}{\delta} \int_{-\delta}^{\delta} \underbrace{|\varphi_\infty(t) - \varphi_n(t)|}_{\to 0}\, dt - 1$$

provided $n \geq N$.

$$\geq 2\left(1 - \frac{\varepsilon}{4}\right) - \frac{\varepsilon}{2} - 1 = 2 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} - 1 = 1 - \varepsilon$$

Hence $\mu_n([-\frac{2}{\delta}, \frac{2}{\delta}]) \geq 1 - \varepsilon$ for $n \geq N$.

This means that for $n \geq N$, $F_n(\frac{2}{\delta}) \geq 1 - \varepsilon$, $F_n(-\frac{2}{\delta}) \leq \varepsilon$.

Let $n \to \infty$, then $F_\infty(\frac{2}{\delta}) \geq 1 - \varepsilon$, $F_\infty(-\frac{2}{\delta}) \leq \varepsilon$.

Thus $F_\infty(x) \to 1$ as $x \to \infty$, $F_\infty(-x) \to 0$ as $x \to -\infty$.

By Helly's Theorem, there is a subsequence $\{\mu_{n_k}\}$ such $\mu_{n_k} \Rightarrow \mu_\infty$ for some distribution μ_∞.

Note first that since $\mu_{n_k} \Rightarrow \mu_\infty$, then $\varphi_{n_k} \to \varphi$ where φ is the ch. fund. μ_∞. But $\varphi_{n_k}(t) \to \varphi_\infty(t)$, hence $\varphi = \varphi_\infty$. Hence φ_∞ is the characteristic function of μ_∞.

There is a metric d on the space of all distribution such that $\mu_n \Rightarrow \mu_\infty$ is equivalent to $d(\mu_n, \mu_\infty) \to 0$ (see Prob. 6.34). Assume for contradiction that $\mu_n \not\Rightarrow \mu_\infty$, then there must be a $\varepsilon > 0$, such that $d(\mu_n, \mu_\infty) > \varepsilon$ for infinitely many n. Make a subsequence ν_m of these n's, i.e. $d(\nu_m, \mu_\infty) > \varepsilon$. Apply the argument above to $\{\nu_m\}$, then we get a subsequence $\{\nu_{m_k}\}$ converging to a limit distribution ν_∞. As before, the char. function of ν_∞ is φ_∞, hence $\nu_\infty = \mu_\infty$.

We now have a strategy for proving convergence in distribution:

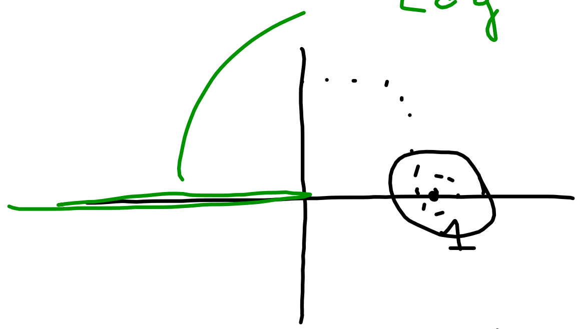
(i) Prove that $\overbrace{\varphi_n(t) \to \varphi_\infty(t)}^{?}$ for some function φ_∞

(ii) Check that φ_∞ is cont. at 0.

Lemma: Assume that $\{z_n\}$ is a sequence of complex numbers converging to z. Then

$$\lim_{n\to\infty} \left(1+\frac{z_n}{n}\right)^n = e^z$$

Log



Proof: $\left(1+\frac{z_n}{n}\right) \to 1$

Note that

$$\mathrm{Log}\left(1+\frac{z_n}{n}\right)^n =$$

$$= n\,\mathrm{Log}\left(1+\frac{z_n}{n}\right) = \frac{\mathrm{Log}\left(1+\frac{z_n}{n}\right)}{\frac{1}{n}}$$

$$\mathrm{Log}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$= \frac{\frac{z_n}{\cancel{n}} \underbrace{- \frac{1}{2}\left(\frac{z_n}{n}\right)^2 + \frac{1}{3}\left(\frac{z_n}{n}\right)^3 - \dots}_{\to 0}}{\frac{1}{\cancel{n}}} \longrightarrow z.$$

Take exponential

$$\left(1+\frac{z_n}{n}\right)^n = e^{n\,\mathrm{Log}\left(1+\frac{z_n}{n}\right)} \to e^z.$$

p 161–62: 6.2, 6.3, 6.4, 6.5, 6.6, 6.8 last year

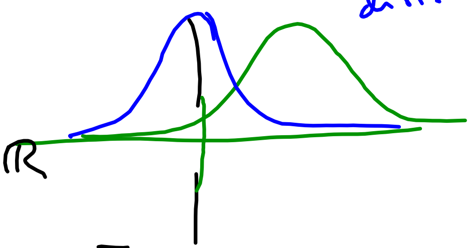
Ex. 6.2: φ is the c.f. for X

$\overline{\varphi}$ — " — $-X$.

$$\varphi_{-X}(t) = E\left[e^{it(-X)}\right] = E\left[\overline{e^{itX}}\right] = \overline{\underbrace{E\left[e^{itX}\right]}_{\varphi(t)}} = \overline{\varphi}(t)$$

Ex 6.3: Show that X has a symmetric dist. if and only φ is real and symmetric.

X and $-X$ has the same dist.



$$\overline{\varphi_X}(t) = \underbrace{\varphi_{-X}(t) = \varphi_X(t)}_{\text{same distr}}, \text{ hence } \varphi_X(t) \in \mathbb{R}$$

$$\varphi(-t) = E\left[e^{i(-t)X}\right] = E\left[e^{it(-X)}\right] = \varphi_{-X}(t)$$

$$= \varphi_X(t). \quad \text{same dist.}$$

6.5: Show that if φ is a c.f., so is $|\varphi|^2$ is a c.f.

$$|\varphi|^2 = \varphi\overline{\varphi}$$

Thinking loud: $\varphi = \varphi_X$

$\overline{\varphi} = \varphi_{-X}$

Pick two independent r.v., both with the dist. of X: X, Y

Then $\varphi_{X-Y}(t) = E\left[e^{it(X-Y)}\right] = E\left[e^{itX} \cdot e^{-itY}\right]$

$$= E\left[e^{itX}\right] E\left[e^{it(-Y)}\right] = \varphi_X(t)\,\varphi_{-Y}(t)$$

$$= \varphi(t)\overline{\varphi}(t) = |\varphi(t)|^2$$

6.6 a) $X_1, X_2, \ldots, X_n$ symmetric $\overset{\text{dist.}}{\vee}$ and independent r.v. Show that $S = X_1 + X_2 + \cdots + X_n$ has sym. dist.

Recall that a dist. is symmetric if and only if φ is real and symmetric.

$$\varphi_S(t) = \varphi_{X_1}(t)\,\varphi_{X_2}(t)\cdots\varphi_{X_n}(t)$$

If all the X's are symmetric, then all the φ's are real and symmetric, then φ_S is real and symmetric.

Counterexample when the X's aren't independent.

$\Omega = \{\omega_1, \omega_2, \omega_3\}$, $P\{\omega_i\} = 1/3$

$X_1(\omega_1) = 1$, $X_1(\omega_2) = -1$, $X_1(\omega_3) = 0$

$X_2(\omega_1) = 1$, $X_2(\omega_2) = 0$, $X_2(\omega_3) = -1$

$$\left.\begin{aligned} P[X_1 + X_2 = 2] &= 1/3 \\ P[X_1 + X_2 = -2] &= 0 \end{aligned}\right\} \text{ not symmetric.}$$

b) $X_1, X_2, \ldots$ i.i.d and $X_i \not\equiv 0$.

Show that with prob 1

$$\left.\begin{aligned} \limsup S_n &= \infty \\ \liminf S_n &= -\infty. \end{aligned}\right\} \text{ tail events.}$$

Assume that this not the case and let

$\hat{S}_n = S_n + \underline{\varepsilon\xi}$ (where ξ is $N(0,1)$ independent of the X's.

Then must be a finite interval (a,b) such that for all n

$$\underline{\varepsilon} < P[\hat{S}_n \in (a,b)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\varphi_{S_n}(t)}\, e^{-\frac{\varepsilon^2 t^2}{2}}\, \frac{e^{-itb} - e^{-ita}}{-2it}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{X_1}(t)^n\, \underbrace{e^{-\frac{\varepsilon^2 t^2}{2}}}_{\text{int.}}\, \underline{\frac{e^{-itb} - e^{-ita}}{-2it}}\, dt \xrightarrow{\text{LDC}} 0.$$

a.s $\downarrow$ 0 except when $|\varphi_{X_1}(t)| = 1$.

$\varphi(t) = E[e^{itX}]$