limsup and liminf for sets

Assume that $\{A_n\}$ is a sequence of events. Then we define

$\limsup A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n\}$

$= \{\omega : \text{For all } k \text{ there is an } n \geq k \text{ such that } \omega \in A_n\}$

$= \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$

$\liminf A_n = \{\omega : \omega \in A_n \text{ for all but a finite number of } n\text{'s}\}$

$= \{\omega : \text{there exists a } k \text{ such that } \omega \in A_n \text{ for all } n \geq k\}$

$= \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} A_n \leftarrow$

Properties: (i) $\liminf A_n \leq \limsup A_n$

(ii) $(\limsup A_n)^c = \liminf (A_n^c)$

(iii) $(\liminf A_n)^c = \limsup (A_n^c)$

Proof (ii) $(\limsup A_n)^c = \left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n\right)^c$

$\overset{D.M.}{=} \bigcup_{k=1}^{\infty} \left(\bigcup_{n \geq k} A_n\right)^c \overset{D.M.}{=} \bigcup_{k=0}^{\infty} \bigcap_{n \geq k} A_n^c = \liminf A_n^c$

A little calculus

Observation: $\lim\limits_{x\to 0} \frac{-\ln(1-x)}{x} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{-\frac{1}{1-x}(-1)}{1} = \underline{\underline{1}}$

Lemma: Assume that $\{x_n\}$ is a sequence of numbers, $0 \le x_n < 1$.
Then the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} -\ln(1-x_n)$ either both converge or both diverge.

Proof: If $x_n \not\to 0$, both series diverge.
If $x_n \to 0$, then we use the Limit Comparison Test:
$\lim\limits_{n\to\infty} \frac{-\ln(1-x_n)}{x_n} = 1$, so they both converge or both diverge.

A few words about products: $\prod\limits_{n=1}^{m} a_n = a_1 a_2 \dots a_m$

$$\prod_{n=1}^{\infty} a_n = \lim_{m\to\infty} \prod_{n=1}^{m} a_n$$

Assume that $\{x_n\}$ is a sequence of numbers $0 \le x_n < 1$ and look at

$$\prod_{n=1}^{\infty}(1-x_n) = \lim_{m\to\infty} \underbrace{\prod_{n=1}^{m}(1-x_n)}_{\substack{\text{decreasing sequence}\\ \text{of positive numbers}}} = \begin{cases} a > 0 \\ 0 \end{cases}$$

Proposition: $\prod\limits_{n=1}^{\infty}(1-x_n) = \lim\limits_{m\to\infty}\prod\limits_{n=1}^{m}(1-x_n) = 0$ iff $\sum\limits_{n=1}^{\infty} x_n = \infty$

Proof: $\prod\limits_{n=1}^{\infty}(1-x_n) = \lim\limits_{m\to\infty}\prod\limits_{n=1}^{m}(1-x_n) = \lim\limits_{m\to\infty}\prod\limits_{n=1}^{m} e^{\ln(1-x_n)}$

$$= \lim_{m\to\infty} e^{\sum_{n=1}^{m}\ln(1-x_n)} \begin{cases} 0 & \text{if } \sum\limits_{n=1}^{\infty}\ln(1-x_n) = -\infty \\ e^{\sum_{n=1}^{\infty}\ln(1-x_n)} > 0 & \text{if } \sum \ln(1-x_n) < \infty \end{cases}$$

$\sum x_n = \infty \Rightarrow \sum\limits_{n=1}^{\infty}\ln(1-x_n) = -\infty$

$\sum x_n < \infty \Rightarrow \sum \ln(1-x_n) < \infty$

Borel-Cantelli Lemma

Borel-Cantelli Lemma: Assume that $\{A_n\}$ is a sequence of events.

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P[\limsup A_n] = 0$.

(ii) Assume in addition that the A_n's are independent. Then

(a) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P[\limsup A_n] = 0$

(b) If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P[\limsup A_n] = 1$.

a 0-1 -law.

Proof (i) $P[\limsup A_n] = P[\bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n\geq k} A_n}_{B_k}]$

Since $\sum_{n=1}^{\infty} P(A_n) < \infty$.

$$\overset{\text{cont of measure}}{=} \lim_{k\to\infty} P[\bigcup_{n\geq k} A_n] \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = 0$$

(ii) Have already proved (a), so we assume that $\sum P(A_n) = \infty$ and prove that $P[\limsup A_n] = 1$.

$$P[\limsup A_n] = 1 - P[(\limsup A_n)^c]$$

$$= 1 - P[\liminf A_n^c] = 1 - P[\bigcup_{n=1}^{\infty} \underbrace{\bigcap_{n\geq k} A_n^c}_{C_k}]$$

$$\overset{\text{cont of measures}}{=} 1 - \lim_{k\to\infty} P(\bigcap_{n\geq k} A_n^c) \overset{\text{independence}}{=} 1 - \lim_{k\to\infty} \prod_{n\geq k} P(A_n^c)$$

$$= 1 - \lim_{k\to\infty} \underbrace{\prod_{n\geq k} (1 - P(A_n))}_{0} = \underline{\underline{1}}$$

$\boxed{\text{Since } \sum P(A_n) = \infty, \ \prod_{n=1}^{\infty} (1-P(A_n)) = 0}$

(Silly) Example: Assume you throw a die infinitely many times. What is the prob. of getting infinitely many 6's?

A_n = throw number n is a six.

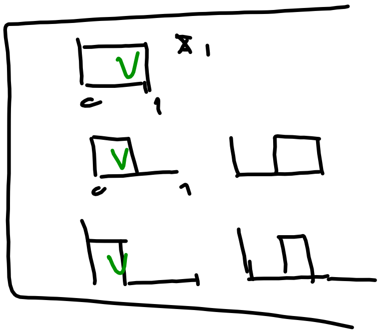
$\limsup A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n\text{'s}\}$

$= \{\omega : \text{we get infinitely many 6's}\}$

But

$$P[\limsup A_n] = 1 \text{ since } \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{6} = \infty.$$

<u>Theorem</u>: Assume that $\{X_n\}$ converges to X in probability. Then there is a subsequence $\{X_{n_k}\}$ that converges to X a.e.



<u>Proof</u>: Since $X_n \to X$ in prob, we know that for every $\varepsilon > 0$, $\underline{P[|X_n - X| > \varepsilon]} \to 0$ as $n \to \infty$.

Choose n_1 such that
$$P[|X_{n_1} - X| > 1] < \frac{1}{2}$$

Choose $n_2 > n_1$ such that
$$P[|X_{n_2} - X| > \tfrac{1}{2}] < \frac{1}{2^2}$$

Choose $n_3 > n_2$ such that
$$P[|X_{n_3} - X| > \tfrac{1}{3}] < \frac{1}{2^3}$$

$\vdots$

Choose $n_k > n_{k-1}$ such that
$$P[|X_{n_k} - X| > \tfrac{1}{k}] < \frac{1}{2^k}.$$

Put $\Lambda_k = \underline{\{\omega : |X_{n_k} - X| > \tfrac{1}{k}\}}$. Then $\underbrace{\sum_{k=1}^{\infty} P(\Lambda_k)}\leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty$.

By the first part of Borel-Cantelli, $P[\limsup \Lambda_k] = 0$.

If we can prove that $X_{n_k}(\omega) \to X(\omega)$ for all $\omega \in (\limsup \Lambda_k)^c$, we shall have convergence a.s

But if $\omega \in (\limsup \Lambda_k)^c = \liminf \Lambda_k^c$
$$= \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} \Lambda_k^c$$
(this means that there must be a n such that for all $k \geq n$, $\omega \in \Lambda_k^c$ i.e. $\underline{|X_{n_k}(\omega) - X(\omega)| \leq \tfrac{1}{k}}$.) and hence $X_{n_k}(\omega) \to X(\omega)$).

Convergence results for expectations

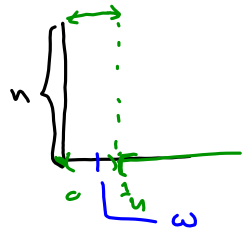
Fundamental question: $X_n \to X$ in some sense

will $E[X_n] \to E[X]$

i.e. $\lim_{n\to\infty} E[X_n] = E[\lim_{n\to\infty} X_n]$

Counterexample: Prob. space $([0,1], \mathcal{B}, P)$ Lebesgue measure.

$$X_n = n \mathbb{1}_{(0,\frac{1}{n})} = \begin{cases} n & \text{if } \omega \in (0,\frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$



Note that $\lim X_n(\omega) = 0$ for all ω.

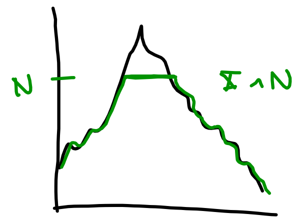
$E[X_n] = 1 = n\cdot\frac{1}{n}$, $E[X] = E[0] = 0$.

$\lim_{n\to\infty} E[X_n] = 1$ $\quad E[\lim_{n\to\infty} X_n] = 0$

Lemma: Assume that X is a r.v. taking values in $[0,\infty]$.
Then $\lim_{N\to\infty} E[X \wedge N] = E[X]$, where $(X\wedge N)(\omega) = \min\{X(\omega), N\}$

Proof: Since $X\wedge N \le X$, we have

$\underbrace{E[X\wedge N]}_{\text{increasing sequence}} \le E[X]$ and $\lim_{N\to\infty} E[X\wedge N] \le E[X]$



Suffices to prove that $\lim_{N\to\infty} E[X\wedge N] \ge E[X]$.

First consider the case where $P[X=\infty] > 0$. Then $E[X] = \infty$ and $\lim E[X\wedge N] = \infty$.

From now on consider the case $P[X=\infty] = 0$.

$$\lim_{N\to\infty} E[X\wedge N] \ge \lim_{N\to\infty} E[X_k \wedge N] \ge \lim_{N\to\infty} \sum_{j=0}^{N\cdot 2^k} \frac{j}{2^k} P[X_k = \frac{j}{2^k}]$$

$$= E[X_k] \to E[X]$$

This shows that $\lim_{N\to\infty} E[X\wedge N] \ge E[X_k]$ for all k!

As $E[X] = \lim E[X_k]$, we get $\lim_{N\to\infty} E[X\wedge N] \ge E[X]$.