Limit theorems for expectations

Lemma: X is a r.v. with values in $[0,\infty]$. Then

$$E[X] = \lim_{N\to\infty} E[X\wedge N]$$

Monotone Convergence Theorem (Strict version) Assume $\{X_n\}$ is an increasing of nonnegative random variables and let $X(\omega) = \lim_{n\to\infty} X_n(\omega)$. Then

$$\lim_{n\to\infty} E[X_n] = E[X] = E[\lim_{n\to\infty} X_n]$$

Proof: Since $X_n \le X$, we have $E[X_n] \le E[X]$ and hence $\lim_{n\to\infty} E[X_n] \le E[X]$. It only remains to prove that

$$\lim_{n\to\infty} E[X_n] \ge E[X]$$

Assume that X is bounded, i.e. $X \le K$ for some $K\in\mathbb{N}$. Given $\varepsilon>0$, let $\Lambda_{\varepsilon,n} := \{\omega : X_n(\omega) \le X(\omega)-\varepsilon\}$. Since $X_n(\omega)\to X(\omega)$, we have $\bigcap_{n\in\mathbb{N}} \Lambda_{\varepsilon,n} = \emptyset$. This is a decreasing intersection, and by continuity of measure

$$\lim_{n\to\infty} P(\Lambda_{\varepsilon,n}) = P[\overset{\emptyset}{\bigcap_{n\in\mathbb{N}} \Lambda_{\varepsilon,n}}] = 0.$$

$$E[X]-E[X_n] = E[X-X_n]$$

$$= \int_{\Lambda_{\varepsilon,n}} \overbrace{(X-X_n)}^{\le K} dP + \int_{\Lambda^c_{\varepsilon,n}} \overbrace{(X-X_n)}^{\le \varepsilon} dP \le KP(\Lambda_{\varepsilon,n}) + \varepsilon \xrightarrow{n\to\infty} \varepsilon$$

Hence $\lim_{n\to\infty} E[X_n] \ge E[X] - \varepsilon$ Since $\varepsilon>0$ is arbitrary, we must have $\lim_{n\to\infty} E[X_n] \ge E[X]$.

Let us now look at the case where X is not (necessarily) bounded:

$$E[X\wedge N] \overset{\text{By what we just proved}}{=} \lim_{n\to\infty} E[X_n\wedge N] \le \lim_{n\to\infty} E[X_n]$$

Hence $\lim E[X_n] \ge \underset{\substack{\downarrow \text{ by lemma.} \\ E[X]}}{E[X\wedge N]}$ for all N. Since this works for all N and $\lim_{N\to\infty} E[X\wedge N] = E[X]$, we must have

$$\lim_{n\to\infty} E[X_n] \ge E[X].$$

Monotone Convergence Theorem (relaxed version): Assume that $\{X_n\}$ is a sequence of nonnegative random variables such that for each n, $X_{n+1} \ge X_n$ a.s. Assume that $X_n \to X$ a.s. Then

$$\lim_{n\to\infty} E[X_n] = E[X].$$

Sketch: There is a null set N such that for $\omega\notin N$, we have $X_{n+1}(\omega) \ge X_n(\omega)$ for all n and $X_n(\omega)\to X(\omega)$. Introduce $\hat{X}_n$ and $\hat{X}$ by

$$\hat{X}_n(\omega) = \begin{cases} X_n(\omega) & \text{for } \omega\notin N \\ 0 & \text{for } \omega\in N \end{cases} \qquad \hat{X}(\omega) = \begin{cases} X(\omega) & \text{for } \omega\notin N \\ 0 & \text{for } \omega\in N. \end{cases}$$

These new random variables satisfy the condition of the "strict version", and hence

$$\begin{array}{ccc} \lim E[\hat{X}_n] & = & E[\hat{X}] \\ \shortparallel & & \shortparallel \\ \lim E[X_n] & = & E[X] \end{array}$$

Fatou's Lemma: Assume that $\{X_n\}$ is a sequence of nonnegative random variables. Then

$$\underline{\liminf E[X_n]} \geq E[\liminf X_n]$$

$\hookrightarrow \liminf_n X_n(\omega)$

Proof: For any, we clearly have that

$$X_n(\omega) \geq \inf_{k \geq n} X_k(\omega)$$

Hence

$$E[X_n(\omega)] \geq E[\inf X_k(\omega)]$$

Take liminf:

$$\underline{\liminf E[X_n]} \geq \liminf E[\overbrace{\inf_{k\geq n} X_k}^{\text{increases with } n}]$$

$$= \lim_{n\to\infty} E[\inf_{k\geq n} X_k]$$

$$\overset{MCT}{=} E[\liminf X_k]$$

(Lebesgue's) Dominated Convergence Theorem: Assume that $\{X_n\}$ is a sequence of random variable converging $X_n \to X$ a.s.

Assume also that there is an integrable random variable Y s.t. $|X_n| \leq Y$ for all n. Then

$$\lim_{n\to\infty} E[X_n] = E[X] = E[\lim_{n\to\infty} X_n]$$

Proof: Apply Fatou's Lemma to $\{Y+X_n\}$.

$$\liminf(E[Y]+E[X_n]) = \liminf E[Y+X_n] \boxed{\geq} E[\overbrace{\liminf (Y+X_n)}^{\nearrow Y+X}] = E[Y+X]$$

$$= \boxed{E[Y]+E[X]}$$

$$\| \quad \boxed{E[Y]+\liminf E[X_n]}$$

Hence

$$E[\cancel{Y}] + \liminf E[X_n] \geq E[\cancel{Y}] + E[X]$$

$$\liminf E[X_n] \geq E[X]$$

Apply Fatou again to $\{Y-X_n\}$:

$$\liminf(E[Y]-E[X_n]) = \liminf E[Y-X_n] \boxed{\geq} E[\liminf (Y-X_n)] = \boxed{E[Y]-E[X]}$$

$$\| \quad E[Y] + \liminf(-E[X_n])$$

$$\| \quad \boxed{E[Y] - \limsup E[X_n]}$$

Hence

$$E[\cancel{Y}] - \limsup E[X_n] \geq E[\cancel{Y}] - E[X]$$

$$\underline{\limsup E[X_n] \leq E[X]}$$

Corollary:

$$\limsup E[X_n] \leq E[X] \leq \liminf E[X_n]$$

Hence

$$\limsup E[X_n] = \liminf E[X_n] = E[X]$$

$$\lim_{n\to\infty} E[X_n] = E[X].$$

Problems

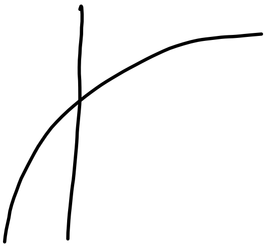
Pages 95-96 3.7, 3.13, 3.18, 3.19
64-65 2.40, 2.41, 2.42, 2.43

3.13: $x_1, x_2, .., x_n$ strictly positive, real numbers:

Arithmetic mean: $A = \frac{x_1+x_2+..+x_n}{n}$

Geometric mean: $G = (x_1 \cdot x_2 \cdots x_n)^{1/n}$

Prove: $G \leq A$



Jensen's Inequality: φ convex, X r.v.

$$\varphi(E[X]) \leq E[\varphi(X)]$$

Need r.v X : $X = x_i$ with prob $\frac{1}{n}$

-"- convex function φ : $\varphi(x) = -\ln(x)$

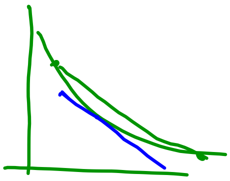
$$-\ln\left(\frac{x_1+x_2+..+x_n}{n}\right) = \varphi(E[X]) \leq E[\varphi(X)] = \Big[-\ln(x_1)\tfrac{1}{n} - \ln(x_2)\cdot\tfrac{1}{n} \; .. - \ln(x_n)\tfrac{1}{n}\Big]$$

$$= -\frac{1}{n}\left[\ln(x_1) + \ln(x_2) + .. + \ln(x_n)\right]$$

$$= -\frac{1}{n}\ln(x_1 x_2 .. x_n) = -\ln\left((x_1 x_2 .. x_n)^{1/n}\right]$$

Hence

$$\ln\left(\frac{x_1+..+x_n}{n}\right) \geq \ln\left((x_1 x_2 . x_n)^{1/n}\right)$$



As ln increasing :

$$\underbrace{\frac{x_1+..+x_n}{n}}_{A} \geq \underbrace{(x_1 x_2 .. x_n)^{1/n}}_{G}.$$

3.19 Minkowski: $\|X\|_2 = E[X^2]^{1/2}$

Show : $\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2$

$\leq 2\, E[X^2]^{1/2} E[Y^2]^{1}$

— Schwarz

Proof: $\|X+Y\|_2^2 = E[(X+Y)^2] = E[X^2] + 2E[XY] + E[Y^2]$

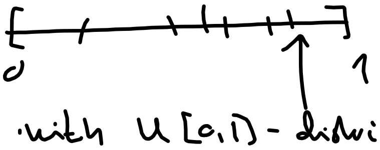
$$\leq \underbrace{E[X^2]}_{} + \underbrace{2E[X^2]^{1/2}E[Y^2]^{1/2}}_{} + \underbrace{E[Y^2]}_{}$$

$$= \Big[\underbrace{(E[X^2])^{1/2}}_{} + \underbrace{(E[Y^2])^{1/2}}_{}\Big]^2 = (\|X\|_2 + \|Y\|_2)^2$$

Hence

$$\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2$$

241 Pick n independent points in the interval [0,1] (uniform distribution). Find expected value of max and min.



$X_1, X_2, .., X_n$ be independent r.v. with $U[0,1]$-distribution

$Z(\omega) = \max\{X_1(\omega), .., X_n(\omega)\}$

Want $E[Z]$?

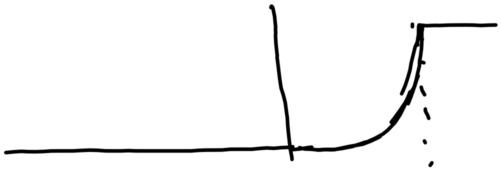
Plan: Find the distribution first:

$$F_Z(x) = P[\,Z \le x\,] = P[\,\max\{X_1, .., X_n\} \le x\,]$$

$$= P[\,X_1 \le x \text{ and } X_2 \le x \text{ and} \ldots \text{ and } X_n \le x\,]$$

$$= P[X_1 \le x]\,P[X_2 \le x] \ldots P[X_n \le x] = x^n$$

Density: $f(x) = F_Z'(x) = n x^{n-1}$



$$E[Z] = \int_0^1 x f(x)\,dx = \int_0^1 x\, n x^{n-1}\,dx$$

$$= n\int_0^1 x^n\,dx = n\left[\frac{x^{n+1}}{n+1}\right]_0^1 = \underline{\underline{\frac{n}{n+1}}}$$

2.42 $X_1, \dots, X_n$ i.i.d random variables with unknown mean μ and variance σ^2.

$$\overline{S}_n(\omega) = \frac{1}{n}\left(X_1(\omega) + X_2(\omega) + \dots + X_n(\omega)\right)$$

$$\frac{1}{\boxed{n-1}} \sum_{i=1}^{n} (X_i - \overline{S}_n)^2 \longleftarrow$$

Show that: $E[\overline{S}_n] = \mu$

$$E\left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{S}_n)^2\right] = \sigma^2$$

$$E[\overline{S}_n] = E\left[\frac{1}{n}[\overset{\mu}{X_1(\omega)} + \dots + \overset{\mu}{X_n(\omega)}]\right) = \mu$$

And

$$E\left[\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{S}_n)^2\right] =$$

$$E\left[\frac{1}{n-1} \sum_{i=1}^{n} ((X_i - \mu) - (\overline{S}_n - \mu))^2\right]$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} E[((X_i - \mu) - (\overline{S}_n - \mu))^2]$$

$$= \frac{n}{n-1} E[((X_1 - \mu) - (\overline{S}_n - \mu))^2] \qquad S_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$= \frac{n}{n-1} E\left[\left((X_1 - \mu) - \frac{(X_1 - \mu) + (X_2 - \mu) + \dots X_n - \mu}{n}\right)^2\right]$$

$$= \frac{n}{n-1} E\left[\left\{\frac{n-1}{n}(X_1 - \mu) - \frac{(X_2 - \mu)}{n} - \dots - \left(\frac{X_n - \mu}{n}\right)\right\}^2\right]$$

$$= \frac{n}{n-1} E\left[\left(\frac{n-1}{n}\right)^2 (X_1 - \mu)^2 + \frac{1}{n^2}(X_2 - \mu)^2 + \dots + \frac{1}{n^2}(X_n - \mu)^2\right]$$

$$= \frac{n}{n-1}\left[\left(\frac{n-1}{n}\right)^2 \sigma^2 + \underbrace{\frac{1}{n^2}\sigma^2 + \dots + \frac{1}{n^2}\sigma^2}_{n-1}\right]$$

$$= \frac{\cancel{n}}{n-1}\left[\left(\frac{n-1}{\cancel{n}}\right)^2 \sigma^2 + \frac{n-1}{n^2}\sigma^2\right]$$

$$= \frac{n-1}{n}\sigma^2 + \frac{1}{n}\sigma^2 = \underline{\underline{\sigma^2}}$$