The Central Limit Theorem

$\{X_n\}$ i.i.d with mean μ and variance σ^2

Laws of large numbers: $\dfrac{X_1+X_2+\cdots+X_n}{n} \to \mu$

Central Limit Theorem: $\dfrac{X_1+X_2+\cdots+X_n - n\mu}{\sqrt{n}} \to N(0,\sigma^2)$

Idea: $\varphi_{\frac{X_1+X_2+\cdots+X_n-n\mu}{\sqrt{n}}} \to$ charact. funct. $N(0,\sigma^2)$

Then Lévy Cont. Th. will give us convergence in distribution

Recall: $\varphi_X(t) = 1 + E[X]it - \frac{1}{2}E[X^2]t^2 + o(t^3)$

$$= 1 + \mu it - \tfrac{1}{2}\sigma^2 t^2 + o(t^3)$$

Central Limit Theorem (i.i.d case) Assume that $\{X_n\}$ is an i.i.d. sequence of random variables with mean μ and variance σ^2. Let $S_n = X_1+X_2+\cdots+X_n$. Then

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \Rightarrow N(0,1)$$

Proof: We may assume that $\mu = 0$ (otherwise just replace X_n by $X_n' = X_n - \mu$). Then

$$\varphi_{X_i}(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^3)$$

Then $\varphi_{\frac{S_n}{\sqrt{n}\sigma}}(t) = E\left[e^{it\frac{S_n}{\sqrt{n}\sigma}}\right] = E\left[e^{i\frac{t}{\sqrt{n}\sigma}(X_1+X_2+\cdots+X_n)}\right]$

$$= E\left[e^{i\frac{t}{\sqrt{n}\sigma}X_1} e^{i\frac{t}{\sqrt{n}\sigma}X_2} \cdots e^{i\frac{t}{\sqrt{n}\sigma}X_n}\right]$$

$$\overset{\text{independence}}{=} E\left[e^{i\frac{t}{\sqrt{n}\sigma}X_1}\right] E\left[e^{i\frac{t}{\sqrt{n}\sigma}X_2}\right] \cdots E\left[e^{i\frac{t}{\sqrt{n}\sigma}X_n}\right]$$

$$= \varphi_{X_1}\left(\frac{t}{\sqrt{n}\sigma}\right)^n = \left(1 - \frac{\sigma^2\left(\frac{t}{\sqrt{n}\sigma}\right)^2}{2} + o\left(\left(\frac{t}{\sqrt{n}\sigma}\right)^3\right)\right)^n$$

$$= \Big(1 \underbrace{- \frac{t^2}{2n} + o\left(\frac{t^3}{n^{3/2}\sigma^3}\right)}_{\frac{z_n}{n}}\Big)^n$$

$\boxed{\text{Recall } \left(1+\frac{z_n}{n}\right)^n \to e^z \text{ if } z_n \to z}$

$\to e^{-\frac{t^2}{2}}$ which is the ch.f of $N(0,1)$

By Lévy's Continuity Theorem $\dfrac{S_n}{\sqrt{n}\sigma} \Rightarrow N(0,1)$.

Central Limit Theorem (Lyapounov's version) Assume that $\{X_n\}$ is a sequence of independent random variables with mean 0 and variance σ_n^2. Assume that we have third moments and put $\gamma_j = E[|X_j|^3]$. Let $S_n = X_1 + X_2 + \dots + X_n$ and $s_n^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$. Assume $\dfrac{\sum_{j=1}^n \gamma_j}{s_n^3} \to 0$ as $n \to \infty$.

Then

$$\frac{S_n}{s_n} \Rightarrow N(0,1) \quad \text{as } n \to \infty.$$

Remarks: (i) Taylor expansion: $f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \frac{1}{6} f'''(c) x^3$

For random variables with mean 0 and variance σ^2

$$\varphi(t) = E[e^{itX}] = 1 + i\,E[X]\,t - \frac{1}{2} E[|X|^2]\,t^2 - \frac{i}{6} E[|X|^3 e^{icX}]\,t^3$$

$$= 1 - \frac{1}{2}\sigma^2 t^2 + \frac{1}{6} t^3 \gamma \Lambda \qquad E[|X|^3]$$

≤ 1 in absolute value

(ii) How reasonable is the condition $\dfrac{\sum_{j=1}^n \gamma_j}{s_n^3} \to 0$?

Check the i.i.d case : $\dfrac{n\gamma}{(s_n^2)^{3/2}} = \dfrac{n\gamma}{(n\sigma^2)^{3/2}} = \dfrac{n\gamma}{n^{3/2}\sigma^3} = \dfrac{\gamma}{n^{1/2}\sigma^3} \to 0$

(iii) $$\left(\frac{\sigma_j^2}{s_n^2}\right)^{3/2} = \frac{E[X_j^2]^{3/2}}{s_n^3} \leq \frac{E[|X_j|^3]}{s_n^3}$$

$$\boxed{\text{Lyapounov's inequality} \quad |E[X]|^p \leq E[|X|^p]}$$

L.E.

$$= \frac{\gamma_j}{s_n^3} \leq \frac{\sum_{j=1}^n \gamma_j}{s_n^3} \to 0$$

Hence $\dfrac{\sigma_j^2}{s_n^2} \to 0$. as $n \to 0$.

Proof: We have

$$\varphi_j(t) = E[e^{itX_j}] = 1 - \frac{t^2\sigma_j^2}{2} + \frac{1}{6}\Delta_j \gamma_j t^3$$ where $|\Delta_j| \le 1$.

$$\varphi_{\frac{S_n}{s_n}}(t) = E[e^{it\frac{S_n}{s_n}}] = E[e^{i\frac{t}{s_n}(X_1 + \dots + X_n)}]$$

$$= \varphi_1\left(\frac{t}{s_n}\right)\varphi_2\left(\frac{t}{s_n}\right)\cdots\varphi_n\left(\frac{t}{s_n}\right)$$

$$= \prod_{j=1}^{n}\Bigg(1 \underbrace{- \frac{\left(\frac{t}{s_n}\right)^2\sigma_j^2}{2} + \frac{1}{6}\Delta_j\gamma_j\left(\frac{t}{s_n}\right)^3}_{\vartheta_{j,n}}\Bigg)$$

$$= \prod_{j=1}^{n}(1 + \vartheta_{j,n})$$

Want to prove $\varphi_{\frac{S_n}{s_n}}(t) \to e^{-\frac{t^2}{2}}$ or equivalently

$$\operatorname{Log}\varphi_{\frac{S_n}{s_n}}(t) \to -\frac{t^2}{2}.$$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$

$$\operatorname{Log}(\varphi_{\frac{S_n}{s_n}}(t)) = \operatorname{Log}\prod_{j=1}^{n}(1+\vartheta_{j,n}) = \sum_{j=1}^{n}\log(1+\vartheta_{j,n})$$

$$= \sum_{j=1}^{n}[\vartheta_{j,n} + \varepsilon(\vartheta_{j,n})]$$ where $|\varepsilon(\vartheta_{j,n})| \le |\vartheta_{j,n}|^2$ for big n

There are two things to prove:

(i) $\sum_{j=1}^{n}\vartheta_{j,n} \to -\frac{t^2}{2}$

(ii) $\sum_{j=1}^{n}\varepsilon(\vartheta_{j,n}) \to 0$ (suffices to prove that $\sum_{j=1}^{n}|\vartheta_{j,n}|^2 \to 0$)

$\frac{\gamma_j}{s_n^3} \le \frac{\sum_{j=1}^{n}\gamma_j}{s_n^3} \to 0$

We get:

(i) $$\sum_{j=1}^{n}\vartheta_{j,n} = \sum_{j=1}^{n}\left(-\frac{\sigma_j^2 t^2}{2s_n^2} + \frac{1}{6}\Delta_j\gamma_j\left(\frac{t}{s_n}\right)^3\right)$$

$$\to -\frac{t^2}{2} + 0 = -\frac{t^2}{2}.$$

bounded by part (i)

(ii) $$\sum_{j=1}^{n}|\vartheta_{j,n}|^2 \le \underbrace{\sup_{j\le n}|\vartheta_{j,n}|}_{\downarrow \text{ need } 0}\sum_{j=1}^{n}|\vartheta_{j,n}|$$

$\le \frac{\gamma_j}{s_n^3} \le \frac{\sum\gamma_j}{s_n^3} \to 0$

$$\sup_{j\le n}|\vartheta_{j,n}| = \sup_{j\le n}\left(-\frac{\sigma_j^2 t^2}{2s_n^2} + \frac{1}{6}\Delta\gamma_j\left(\frac{t}{s_n}\right)^3\right) \to 0.$$

$\to 0$ as $n \to \infty$

Hence $\sum_{j=1}^{n}|\vartheta_{j,n}|^2 \to 0$, and we are done.

$\frac{\sigma_j^2}{s_n^2} \to 0$.

Stochastic Processes (7.1) + (7.4)

Let $(\Omega, \mathcal{F}, P)$ be a prob. space. and let $\mathbb{T} \subseteq \mathbb{R}$.

A stochastic process on $(\Omega, \mathcal{F}, P)$ indexed by $\mathbb{T}$ is just a collection $\{X_t\}_{t \in \mathbb{T}}$ of random variables on $(\Omega, \mathcal{F}, P)$.

Typical cases:

(i) $\mathbb{T} = \mathbb{N}$
(ii) $\mathbb{T} = \mathbb{Z}$.
(iii) $\mathbb{T} = \{1, 2, 3, \ldots, N\}$

} discrete time processes

(iv) $\mathbb{T} = [0, \infty)$
(v) $\mathbb{T} = \mathbb{R}$
(vi) $\mathbb{T} = [a, b]$

} continuous time process.

Fix an $\omega \in \Omega$ (fix one scenario). Then $t \mapsto X_t(\omega)$ is called a path of the process.

Example: Toss a coin N times and call the results 1 for heads and -1 for tails.

Basic event: $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ where each $\omega_i = \pm 1$.

$$\Omega = \{\omega : \omega = (\omega_1, \omega_2, \ldots, \omega_N)\}.$$

$$S_n(\omega) = \omega_1 + \omega_2 + \cdots + \omega_n$$ (our gain at time n given that ω is what happens)

Fix ω, and look at the path.