Stopping times

Time line: $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

Assume that $(\Omega, \mathcal{F}, P)$ is a probability space. A filtration on $(\Omega, \mathcal{F}, P)$ is a sequence $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ of σ-algebras such that

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n \subseteq \dots \subseteq \mathcal{F}.$$

Idea: $\mathcal{F}_n$ encodes the information at time n.

Example: If $\{X_n\}_{n \in \mathbb{N}_0}$ is a stochastic process

$$\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$$

Definition: A stochastic process X is adapted to the filtration $\{\mathcal{F}_n\}$ is X_n is $\mathcal{F}_n$-measurable for all n.

Definition: A stopping time T is a function $T : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ such that for all $n \in \mathbb{N}_0$,

$$\underbrace{[T = n]}_{\{\omega : T(\omega) = n\}} \in \mathcal{F}_n.$$

Prop: T is a stopping time iff

$$[T \leq n] \in \mathcal{F}_n \text{ for all } n.$$

Proof: Assume that T is a stopping time.

$$[T \leq n] = \underset{\mathcal{F}_1}{\underset{\cap}{[T = 1]}} \cup \underset{\mathcal{F}_2}{\underset{\cap}{[T = 2]}} \cup \dots \cup \underset{\mathcal{F}_n}{\underset{\cap}{[T = n]}} \in \mathcal{F}_n.$$

Assume next that $[T \leq n] \in \mathcal{F}_n$ for all n.

Then

$$[T = n] = \underbrace{[T \leq n]}_{\mathcal{F}_n} \setminus \underbrace{[T \leq n-1]}_{\mathcal{F}_{n-1} \subseteq \mathcal{F}_n} \in \mathcal{F}_n.$$

Examples (i) $T = K$ (constant) for all ω.

$$[T=n] = \left\{ \begin{array}{ll} \phi & \text{if } n \neq K \\ \Omega & \text{if } n = K \end{array} \right\} \mathcal{F}_n.$$

(ii) First hitting time: X_n is an adapted process. Choose a Borel set $B \subseteq \mathbb{R}$, and let

$T(\omega)$ = the first n such that $X_n(\omega) \in B$

($= \infty$ if no such n exists)

T is a stopping time:

$$[T \leq n] = \underbrace{[X_1 \in B]}_{\mathcal{F}_1 \subseteq \mathcal{F}_n} \cup \underbrace{[X_2 \in B]}_{\mathcal{F}_2 \subseteq \mathcal{F}_n} \cup \dots \cup \underbrace{[X_n \in B]}_{\mathcal{F}_n} \in \mathcal{F}_n.$$

Prop: Assume that S, T are two stopping times. Then

$(S \wedge T)(\omega) = \min\{S(\omega), T(\omega)\}$, $(S \vee T)(\omega) = \max\{S(\omega), T(\omega)\}$

are stopping times.

Proof. $[S \wedge T \leq n] = [S \leq n] \cup [T \leq n]$

$[S \vee T \leq n] = [S \leq n] \cap [T \leq n]$.

Def: Assume that X is a stochastic process $X = \{X_n\}_{n \in \mathbb{N}_0}$ and that T is a stopping time taking only finite values. Then the stopped process X_T is the random variable Y

$$Y(\omega) = X_{T(\omega)}(\omega).$$

If T is a stopping time, then

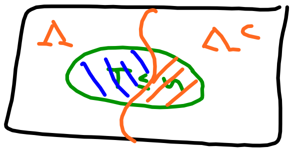
$$\mathcal{F}_T = \{\Lambda \in \mathcal{F} : \underline{\Lambda \cap [T \le n]} \in \mathcal{F}_n \text{ for all } n\}$$

<u>Observation:</u> We can replace $[T \le n]$ by $[T = n]$ in this def.

$$\mathcal{F}_T = \{\Lambda \in \mathcal{F} : \Lambda \cap [T = n] \in \mathcal{F}_n \text{ for all } n\}$$

<u>Observation:</u> If $T(\omega) = N$ for all ω, then $\mathcal{F}_T = \mathcal{F}_N$

<u>Prop:</u> $\mathcal{F}_T$ is a σ-algebra



<u>Proof:</u> Must check the def. of a σ-algebra:

(i) $\phi \in \mathcal{F}_T$: $\phi \cap [T \le n] = \phi \in \mathcal{F}_n$

(ii) $\Lambda \in \mathcal{F}_T \Rightarrow \Lambda^c \in \mathcal{F}_T$: $\Lambda^c \cap [T \le n] = \underset{\in \mathcal{F}_n}{[T \le n]} \setminus \underset{\in \mathcal{F}_n}{([T \le n] \cap \Lambda)} \in \mathcal{F}_n$

(iii) $\Lambda_k \in \mathcal{F}_T$ for all $k \Rightarrow \bigcup_{k \in \mathbb{N}} \Lambda_k \in \mathcal{F}_T$.

$$\left(\bigcup_{k \in \mathbb{N}} \Lambda_k\right) \cap [T \le n] = \bigcup_{k \in \mathbb{N}} \underbrace{(\Lambda_k \cap [T \le n])}_{\in \mathcal{F}_n} = \mathcal{F}_n.$$

<u>Prop:</u> T is $\mathcal{F}_T$-measurable

<u>Proof:</u> Must show that $[T \le k] \in \mathcal{F}_T$ for all k.

$$[T \le k] \cap [T = n] = \begin{cases} \phi & k < n \\ [T = n] & k \ge n \end{cases} \in \mathcal{F}_n$$

<u>Prop:</u> If S, T are two stopping times and $S \le T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$

<u>Proof:</u> Assume $\Lambda \in \mathcal{F}_S$, must prove that $\Lambda \in \mathcal{F}_T$

$$\Lambda \cap [T \le n] = (\overbrace{\Lambda \cap \underline{[S \le n]}}^{\mathcal{F}_n}) \cap \overbrace{\underline{[T \le n]}}^{\mathcal{F}_n} \in \mathcal{F}_n, \text{ hence } \Lambda \in \mathcal{F}_T.$$

Problems

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6.9 Distribution F, char. funct. φ

If $\underbrace{\varphi \text{ is integrable}}$, then F has a continuous density.

$$\int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty.$$

Lévy's Inversion Formula:

$$\bar{F}(b) - \bar{F}(a) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-2\pi i t} \varphi(t)\, e^{-\frac{\varepsilon^2 t^2}{2}} dt$$

$e^{-\frac{\varepsilon^2 t^2}{2}} \to 1$

Provided φ is integrable

$$= \int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-2\pi i t} \varphi(t)\, dt$$

$$\frac{\bar{F}(x+h) - \bar{F}(x)}{h} = \int_{-\infty}^{\infty} \frac{e^{-i(x+h)t} - e^{-ixt}}{-2\pi i t h} \varphi(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \left[\frac{e^{-iht} - 1}{-iht}\right] \varphi(t)\, dt$$

$\cos 0$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \frac{(\cos(-ht) - 1) + i \sin(-ht)}{-iht} \varphi(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \left\{\left[\frac{\overset{\cos 0}{1} - \cos(-ht)}{i(-ht)}\right] + \left[\frac{\overset{\sin 0}{0} - \sin(-ht)}{-ht}\right]\right\} \varphi(t)\, dt$$

MVT MVT

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \left[\underbrace{-\frac{\sin(-hc)}{i}}_{\to 0} \cdot + \underbrace{\cos(-hd)}_{\to 1} \right] \varphi(t)\, dt$$

$$\lim_{h\to 0} \frac{\overset{F}{\bar{F}}(x+h) - \overset{F}{\bar{F}}(x)}{h} = \lim_{h\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \left[i \underline{\sin(-hc)} + \underline{\cos(-hd)} \right] \varphi(t)\, dt$$

$$\overset{\text{LDCT}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \lim_{h\to 0} \left[i \overset{0}{\sin(-hc)} + \overset{1}{\cos(-hd)} \right] \varphi(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \varphi(t)\, dt.$$

This shows that $F(x)$ is differentiable with derivative

$$F'(x) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} \varphi(t)\, dt}_{f(x)\ -\ \text{continuous}}.$$

6.21 a) If $X_n \Rightarrow X$, then $E[f(X_n)] \to E[f(X)]$
for all bounded, continuous functions f.

Does this still hold for all bounded, Borel functions f.

No! $f = \mathbb{1}_{[0,\infty)}$ $X_n = -\frac{1}{n}$, $X = 0$.

$X_n \Rightarrow X$

$E[f(X_n)] = E[0] = 0$

$\ne$

$E[f(X)] = E[1] = 1$

b) Weak convergence $E[f(X_n)] \to E[f(X)]$ for all bounded cont. funct

Vague —— " —— for all bounded cont. funct. with compact.

$X_n(\omega) = n$ $\quad E[\underbrace{f(X_n)}_{0}] \to 0$

$\Downarrow$

$\mu_n \xRightarrow{\text{vague}} \mu_0$ $\quad f = \mathbb{1}$

$E[f(X_n)] = 1$

6.22 a) $X_n \to X$ in prob
$Y_n \to Y$ in prob

Show that $X_n + Y_n \to X + Y$ in dist.

b) Find X_n, Y_n ~~such~~ X, Y such that
$X_n \to X$ in dist
$Y_n \to Y$ in dist.
But
$X_n + Y_n \not\to X + Y$ in dist.

a) Need to prove that

$$E[f(X_n+Y_n)] \to E[f(X+Y)] \text{ for all bounded cont. } f.$$

Pick subsequence $\{X_{n_k}\}, \{Y_{n_k}\}$ converging a.e. to X and Y.
The $f(X_{n_k}+Y_{n_k}) \to f(X+Y)$ a.e.

LDCT:

$$\lim_{k\to\infty} E[f(X_{n_k}+Y_{n_k})] = E[f(X+Y)]$$

Assume that this does not hold for the original sequence.
Then there is an $\varepsilon > 0$ and a subsequence X_{m_k}, Y_{m_k} such that

$$|E[f(\underbrace{X_{m_k}+Y_{m_k}}))] - E[f(X+Y)]| > \varepsilon.$$

Further subsequences converge pointwise, contradiction.

b) Counterexample $X_n \Rightarrow X$, $Y_n \Rightarrow Y$

$X_n + Y_n \not\Rightarrow X + Y = 2X$

The trouble is independence
X_n, Y_n independent r.v.s of the same distribution.
$X = Y$.