Limit theorems for expectations

Recall:

Convergence in prob.
$\Downarrow$

Lebesgue's Dominated Convergence Theorem: If X_n converges a.s. to X and there is an integrable r.v. Y such that $|X_n| \leq Y$ a.s. for all n, then

$$\lim_{n\to\infty} E[X_n] = E[X]$$

Lemma: Assume that $\{x_n\}$ is a bounded sequence of real numbers with a subsequence $\{y_n\}$ converging to a number a. If $\{x_n\}$ does not converge to a, then $\{x_n\}$ has another subsequence $\{z_n\}$ converging to a number $b \neq a$.

Proof: Assume that $\{x_n\}$ does not converge to a. There must be a $\varepsilon > 0$ such that $|x_n - a| \geq \varepsilon$ for arbitrarily large n. Hence there is a subsequence $\{u_n\}$ of $\{x_n\}$ such that $|u_n - a| \geq \varepsilon$ for all n. Since $\{u_n\}$ is bounded it has a convergent subsequence $\{z_n\}$ converging to a number b. Since $|z_n - a| \geq \varepsilon$ for all n, $b \neq a$.

DCT for convergence in prob.: Assume $\{X_n\}$ is converging to X in probability and that there is an integrable r.v. Y such that $|X_n| \leq Y$. Then

$$\lim_{n\to\infty} E[X_n] = E[X]$$

Proof: We know that $\{X_n\}$ has a subsequence $\{Y_n\}$ converging to X a.s. Hence

$$\lim_{n\to\infty} E[Y_n] = E[X].$$

Assume for contradiction that $E[X_n]$ does not converge to $E[X]$. By the lemma, there is another subsequence $\{E[Z_n]\} \to b$ converging to a $b \neq E[X]$.
Now since Z_n converges to X in prob., Z_n has a subsequence $\{U_n\}$ converging to X a.s. By the original form of the DCT, we get $E[U_n] \to E[X]$.
But $\{E[U_n]\}$ is a subsequence of $E(Z_n) \to b$, and $E(U_n) \to b \neq E[X]$. This is a contradiction as $E[U_n]$ cannot both converge to $E[X]$ and $b \neq E[X]$.
Hence $E[X_n] \to E[X]$ QED.

Observation: As we are working with prob. spaces, all constant functions are integrable, and we may often use a constant Y in DCT. This is often called the Bounded C. T.

Corollary: Assume that $\{X_n\}$ converges to X either a.s. or in probability. Assume also that there is an integrable Y such that $|X_n| \leq Y$ a.s. Then $X_n \to X$ in L^1-norm

i.e $$E[|X - X_n|] \to 0.$$

Proof: Note that $|X - X_n| \leq |X| + |X_n| \leq 2|Y|$ a.s.

Applying the DCT, we get

$$\lim_{n\to\infty} E[\overbrace{\underset{\downarrow \atop 0}{|X - X_n|}}^{\leq 2|Y|}] = E[0] = 0$$

Laws of large number

Idea: If $\{X_n\}$ is a sequence of independent, identically distributed r.v. with mean μ, then

$$\frac{X_1+X_2+\cdots+X_n}{n} \longrightarrow \mu$$

$X_n - \mu$

in what sense

Assume that $E[X_i]=0$:

Weak laws of large numbers

Conditions $\Rightarrow$ $\dfrac{X_1+\cdots+X_n}{n} \to 0$ in prob.

Strong laws of large numbers

Conditions $\Rightarrow$ $\dfrac{X_1+\cdots+X_n}{n} \to 0$ a.s.

(A) Weak law of large numbers: Assume that $\{X_n\}$ is an independent sequence of random variable with $E[X_n]=0$ and $E[X_n^2] \le \sigma^2$ for all n. Then

$$\frac{X_1+\cdots+X_n}{n} \to 0 \text{ in prob.}$$

Proof: Let $S_n = X_1+X_2+\cdots+X_n$, must prove $\frac{S_n}{n} \to 0$ in prob.
By Chebyshev's inequality with $p=2$:

$$P\left[\left|\frac{S_n}{n}\right| \ge \varepsilon\right] \le \frac{1}{\varepsilon^2} E\left[\left(\frac{S_n}{n}\right)^2\right]$$

$$= \frac{1}{n^2\varepsilon^2} E\left[(X_1+X_2+\cdots+X_n)^2\right]$$

$$= \frac{1}{n^2\varepsilon^2} E\left[X_1^2+X_2^2+\cdots+X_n^2\right]$$

$$\le \frac{1}{n^2\varepsilon^2} n\sigma^2 = \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

Hence $\frac{S_n}{n} \to 0$ in prob.

Strong law of large numbers

Observation: Assume $E[X^4] < \infty$. Then

$$E[X^2] \leq E[X^4]^{1/2}$$

since by Schwarz' inequality

$$E[X^2] = E[1 \cdot X^2] \leq \underbrace{(E[1^2])}_{1}^{1/2} E[(X^2)^2]^{1/2} = E[X^4]^{1/2}$$

Cantelli's strong law of large numbers: Assume that $\{X_n\}$ is an independent sequence of r.v. with mean 0 and $E[X_n^4] \leq M$ for all n. Then

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \to 0 \quad \text{a.s.}$$

Proof: $E[S_n^4] = E[(X_1 + X_2 + \cdots + X_n)^4]$

$$E\Big[\sum_{i,j,k,l} X_i X_j X_k X_l\Big] = \sum_{i,j,k,l} E[X_i X_j X_k X_l]$$

X_i different

$$E[X_i X_j X_k X_l] = \underbrace{E[X_i]}_{0} E[X_j X_k X_l]$$

$$= \sum_{i=1}^{n} E[X_i^4] + 3\sum_{i \neq j} E[X_i^2 X_j^2]$$

$$= \underbrace{\sum_{i=1}^{n} \overset{\leq M}{E[X_i^4]}}_{n} + 3 \underbrace{\sum_{i \neq j} E[X_i^2]E[X_j^2]}_{n\,M}$$

$M^{1/2} \cdot M^{1/2} = M$

$$E[X_i^2] \leq E[X_i^4]^{1/2} \leq M^{1/2}$$

$$\leq nM + 3n(n-1)M$$

$$= nM + 3n^2M - 3nM \leq 3n^2 M$$

Hence: $\underline{E[S_n^4] \leq 3n^2 M}$.

Chebyshev's inequality with $p = 4$ and $\lambda = \frac{1}{n^{1/8}}$:

$$\boxed{P[|X| \geq \lambda] \leq \frac{1}{\lambda^p} E[|X|^p]}$$

$$P\Big[\Big|\frac{S_n}{n}\Big| \geq \frac{1}{n^{1/8}}\Big] \leq \frac{1}{\left(\frac{1}{n^{1/8}}\right)^4} E\Big[\Big|\frac{S_n}{n}\Big|^4\Big] \qquad \left(\frac{1}{n^{1/8}}\right)^4 = \frac{1}{n^{1/2}}$$

$$= n^{1/2} \frac{1}{n^4} E[|S_n|^4] \leq \frac{n^{1/2}}{n^4} 3n^2 M$$

$$(2 + \tfrac{1}{2}) - 4 = -\tfrac{3}{2}$$

$$= \frac{3M}{n^{3/2}}, \quad \text{Calculus: } \sum \frac{1}{n^{3/2}} < \infty.$$

Borel-Cantelli now tells us that

$$P\Big[\limsup_{n \to \infty} \Big\{\omega : \Big|\frac{S_n(\omega)}{n}\Big| \geq \frac{1}{n^{1/8}}\Big\}\Big] = 0$$

For all ω outside this set of prob. 0, we can only have $\frac{|S_n(\omega)|}{n} \geq \frac{1}{n^{1/8}}$ for finitely many n. Hence for sufficiently large $\frac{|S_n(\omega)|}{n} < \frac{1}{n^{1/8}} \to 0$.

Thus $\Big|\frac{S_n(\omega)}{n}\Big| \to 0$ on a set of prob 1, hence

Sect 5.4: σ-algebras as information

Prob space: $(\Omega, \mathcal{F}, P)$

Assume that $\mathcal{G}$ is another σ-algebra on Ω (usually $\mathcal{G} \subseteq \mathcal{F}$)

A function $X : \Omega \to \mathbb{R}$ is called $\mathcal{G}$-measurable if

$$\{\omega : X(\omega) \leq x\} \in \mathcal{G} \text{ for all } x.$$

Prop.: If X is $\mathcal{G}$-measurable, then $X^{-1}(B) \in \mathcal{G}$ for all Borel sets B.

Definition: If $X : \Omega \to \mathbb{R}$, we define the σ-algebra $\sigma(X)$ generated by X to be the smallest σ-algebra such that $\{\omega : X(\omega) \leq x\} \in \sigma(X)$ (in other words, $\sigma(X)$ is the σ-algebra generated by the family $\{\{\omega : X(\omega) \leq x\}\}_{x \in \mathbb{R}}$.)

Prop: X is measurable w.r.t. $\sigma(X)$.

Definition. If $\{X_i\}_{i \in I}$ is a family of function $X_i : \Omega \to \mathbb{R}$, then the σ-algebra $\sigma(\{X_i\}_{i \in I})$ generated by $\{X_i\}_{i \in I}$ is the smallest σ-algebra containing all sets

$$\{\omega : X_i(\omega) \leq x\}$$

for all $i \in I$ and all $x \in \mathbb{R}$.