0-1-laws

Borel-Cantelli Lemma, part II: If $\{A_n\}$ is independent, then $P[\limsup A_n]$ is either 0 or 1

Assume that $\{X_n\}$ is a sequence of ind. r.v.

Past σ-algebras: $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$

Future σ-algebras: $\mathcal{F}_n^* = \sigma(X_{n+1}, X_{n+2}, \ldots)$

Note that $\mathcal{F}_n$ and $\mathcal{F}_n^*$ are independent.

σ-algebra at infinity: $\mathcal{F}_\infty^* = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n^*$

An element in $\mathcal{F}_\infty^*$ is called a tail event.

Examples: $\limsup X_n \in A$ is a tail event

$\{\omega : \sum X_n(\omega) \text{ converges}\}$ — " —

$\{\omega : \sum X_n(\omega) \in A\}$ is usually not a tail event.

Kolmogorov/Borel 0-1 law: If $\{X_n\}$ is an independent sequence of random variables and Λ is a tail event (i.e. $\Lambda \in \mathcal{F}_\infty^*$), then $P(\Lambda)$ is either 0 or 1.

$x^2 = x$

Proof: Let

$$\mathcal{H} = \{\Gamma : P(\Gamma \cap \Lambda) = P(\Gamma)P(\Lambda)\}.$$

Note that $\mathcal{F}_n \subseteq \mathcal{H}$ because if $\Gamma \in \mathcal{F}_n$, then Γ is independent of Λ since $\Lambda = \mathcal{F}_\infty^* = \bigcap_{k=1}^{\infty} \mathcal{F}_k^* \subseteq \mathcal{F}_n^*$. and $\mathcal{F}_n$ is independent of $\mathcal{F}_n^*$. Thus

$$\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subseteq \mathcal{H}$$

Now $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an algebra, and hence

$$\sigma\Big(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\Big) = \mathcal{M}\Big(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\Big).$$ (the smallest monotone class containing $\bigcup \mathcal{F}_n$)

If we can prove that $\mathcal{H}$ is a monotone class, then

$$\sigma\Big(\bigcup \mathcal{F}_n\Big) = \mathcal{M}\Big(\bigcup \mathcal{F}_n\Big) \subseteq \mathcal{H}.$$

Check that $\mathcal{H}$ is a monotone class.

(i) Let $\{\Gamma_n\}$ be an increasing sequence in $\mathcal{H}$. We must show that $\bigcup_{n\in\mathbb{N}} \Gamma_n \in \mathcal{H}$.

$$P\Big[\Big(\bigcup_{n\in\mathbb{N}}\Gamma_n\Big) \cap \Lambda\Big] = P\Big[\overbrace{\bigcup_{n\in\mathbb{N}}(\Gamma_n \cap \Lambda)}^{\text{increasing}}\Big] \overset{\text{cont meas}}{=} \lim_{n\to\infty} P[\Gamma_n \cap \Lambda]$$

$$\overset{\Gamma_n \in \mathcal{H}}{=} \lim_{n\to\infty} \underbrace{P(\Gamma_n)}_{\text{increa}} P(\Lambda) \overset{\text{cont meas}}{=} P\Big(\bigcup_{n\in\mathbb{N}}\Gamma_n\Big)P(\Lambda).$$

(ii) Let now $\{\Gamma_n\}$ be a decreasing sequence in $\mathcal{H}$:

$$P\Big(\bigcap_{n\in\mathbb{N}}\Gamma_n \cap \Lambda\Big) = P\Big(\bigcap_{n\in\mathbb{N}}(\Gamma_n \cap \Lambda)\Big) = \lim_{n\in\mathbb{N}} P(\Gamma_n \cap \Lambda)$$

$$= \underbrace{\lim_{n\to\infty} P(\Gamma_n)}P(\Lambda) = P\Big(\bigcap_{n\in\mathbb{N}}\Gamma_n\Big)P(\Lambda).$$

This establishes that $\mathcal{M}$ is a monotone class and hence

$$\sigma\Big(\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\Big) \subseteq \mathcal{H}.$$

Since $\Lambda \in \mathcal{F}_\infty^* = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n^* \subseteq \sigma\Big(\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\Big) \subseteq \mathcal{H}$.

Thus Λ is independent of itself and

$$P(\Lambda) = P(\Lambda \cap \Lambda) = P(\Lambda)P(\Lambda) = P(\Lambda)^2$$

Complex-valued random variables

$Z = X + iY$, where X and Y are real valued random variables.

Z is integrable when both X and Y are, and we then define the expectation by

$$E[Z] = E[X] + i\,E[Y]$$

Prop: Z is integrable iff $|Z|$ is integrable.

Prop: If $\alpha, \beta \in \mathbb{C}$ and Z, W are integrable, complex valued random variables, then

$$E[\alpha Z + \beta W] = \alpha E[Z] + \beta E[W]$$

Prop: If Z is integrable, then

$$|E[Z]| \leq E[|Z|]$$

Proof: Since $E[Z]$ is a complex number, there is $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha E(Z) = |E(Z)|$

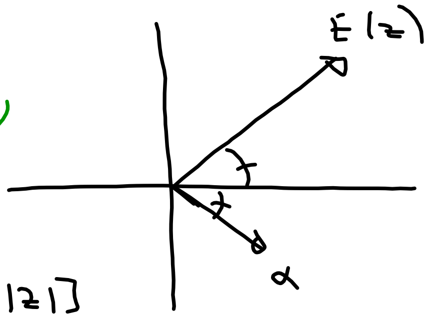
Then

$$|E(Z)| = \alpha E(Z) = E(\alpha Z)$$

$$= E[U + iV] = E[U] + i\,E[V] \qquad 0$$

$$\leq E[|U|] \leq E[|\alpha Z|] = E[|\alpha|\,|Z|]$$

$$= E[|Z|]$$



Characteristic function

If X is a random variable, we define its characteristic function

$$\varphi(t) = E[e^{itX}] = E[\cos tX + i \sin tX]$$
$$= E[\cos tX] + i\,E[\sin tX]$$

Compare the moment generating function

$$M(t) = E[\underbrace{e^{tX}}_{\text{integrability problems.}}]$$

Prop: $\varphi(0) = 1$, $|\varphi(t)| \leq 1$.

Proof: $\varphi(0) = E[e^{i0X}] = E[\underbrace{e^0}_{1}] = 1$.

$$|\varphi(t)| = |E[e^{itX}]| \leq E[|e^{itX}|] = E[1] = 1$$

Prop: φ is uniformly continuous.

Proof: $|\varphi(t+s) - \varphi(t)| = |E[e^{i(t+s)X} - e^{itX}]|$

$$\leq E[|\underbrace{e^{itX}}_{\text{abs val. } 1}(e^{isX}-1)|] = E[\overbrace{|e^{isX}-1|}^{\to 0}] = \eta(s)$$

$\underset{\leq 2}{}$

If we can prove that $\eta(s) \to 0$ as $s \to 0$, then φ must be uniformly continuous. Why? Given an $\varepsilon > 0$, we can choose δ so small that $|\eta(s)| < \varepsilon$ for all $|s| < \delta$. But this means that if $|s| < \delta$, then for all t, $|\varphi(t+s) - \varphi(t)| < \eta(s) < \varepsilon$ when $|s| < \delta$.

So how do we prove that $\eta(s) \to 0$? Assume for contradiction that $\eta(s) \not\to 0$. Then there is a sequence $\{s_n\} \to 0$ such that $\eta(s_n) \not\to 0$. But

$$\eta(s_n) = E[\overbrace{|e^{is_nX} - 1|}^{\to 0}] \xrightarrow{DCT} E[0] = 0.$$

$\underset{\leq 2}{}$

Contradiction, so $\eta(s) \to 0$. QED.

Prop: φ is a positive define function, i.e. for all real numbers $t_1, t_2, \dots, t_n$ and all complex numbers $z_1, z_2, \dots, z_n$, then
$$\sum_{i,j=1}^{n} \varphi(t_i - t_j) z_i \overline{z_j} \geq 0.$$

$M_{ij} = \varphi(t_i - t_j)$

M is a positive matrix

Prove: $\sum_{i,j=1}^{n} \varphi(t_i - t_j) z_i \overline{z_j}$

$$= \sum_{i,j=1}^{n} E[\overline{e^{i(t_i - t_j)X}}] z_i \overline{z_j}$$

$$= E\Big[\sum_{i,j=1}^{n} e^{it_iX} \left(\overline{e^{it_jX}}\right) z_i \overline{z_j}$$

$$= E\Big[\sum_{i,j=1}^{n} (z_i e^{it_iX}) (\overline{z_j e^{it_jX}})\Big]$$

$$= E\Big[\Big(\sum_{i=1}^{n} z_i e^{it_iX}\Big) \Big(\overline{\sum_{j=1}^{n} z_j e^{it_jX}}\Big)\Big]$$

$$E\Big[\Big|\sum_{i=1}^{n} z_i e^{it_iX}\Big|^2\Big] \geq 0.$$

Theorem: Assume that $E[|X|^k] < \infty$. Then φ is k times differentiable and
$$\varphi(t) = \sum_{j=0}^{k} \frac{1}{j!} E[X^j] (it)^j + \underbrace{o(t^k)}$$

In particular:
$$\varphi^{(j)}(0) = i^j E[X^j]$$

Idea behind the proof:

$$\varphi'(t) = \frac{d}{dt} E[e^{itX}] \overset{?}{=} E\Big[\frac{d}{dt} e^{itX}\Big]$$

$$= E[iX e^{itX}] = i E[X e^{itX}]$$

$$\varphi^{(j)}(t) = i^j E[X^j e^{\overset{t=0 \to 1}{itX}}] \qquad j \leq k.$$

Taylor expansion of order k:
$$\varphi(t) = \sum_{j=0}^{k} \frac{\varphi^{(j)}(0)}{j!} t^j + o(t^k)$$

$$= \sum_{j=0}^{n} \frac{E[X^j]}{j!} (it)^j + o(t^k)$$ — formula in theorem.

We need to check that we can pull $\frac{d}{dt}$ inside the expectation.

$$\varphi'(t) = \lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \lim_{h \to 0} \frac{1}{h} E[e^{i(t+h)X} - e^{itX}]$$

$$= \lim_{h \to 0} E\Big[e^{itX} \frac{e^{ihX} - 1}{h}\Big]$$

$$= \lim_{h \to 0} E\Big[e^{itX}\Big[\frac{\cosh X - \overset{\cos 0}{1}}{hX} X + i \frac{\overset{-\sin(0)}{\sin hX}}{hX} X\Big]\Big]$$

$$= \lim E\Big[e^{itX}\Big[\underbrace{\overbrace{-\sin \hat{h} X \cdot X}^{\leq 2|X|}}_{\downarrow\, 0} + \underbrace{\overbrace{i \cos \hat{h} X \cdot X}}_{\downarrow\, 1}\Big]\Big]$$

$$\overset{DCT}{=} E[e^{itX} iX] = i E[X e^{itX}]$$

Higher order derivative by induction in a similar manner.