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Padlet:

Wednesday: 2 lectures

Thursday 1 lecture + 1 problems

Prereq.: Calculus + lin. alg. | MAT 1100 + 1110 + 1120 |

Prob. STK 1100

Advantages: { STK 2130

MAT 2400

MAT 3400

Prob. th. based on measure theory.

Recalling set theory

A set is a collection (finite or infinite) of mathematical objects.

$a \in A$, $a \notin A$

$A \subseteq B$ "a subset of B"

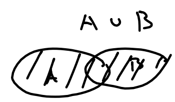
Boolean operations:

Union: $A_1 \cup A_2 \cup \dots \cup A_n = \{a : a \in A_i \text{ for at least one } i\}$

$\bigcup_{n \in \mathbb{N}} A_n = \{a : a \in A_n \text{ for at least one } n\}$

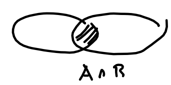
Intersection: $A_1 \cap A_2 \cap \dots \cap A_n = \{a : a \in A_i \text{ for all } i\}$

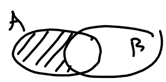
$\bigcap_{n \in \mathbb{N}} A_n = \{a : a \in A_n \text{ for all } n\}$



Set-theoretic difference:

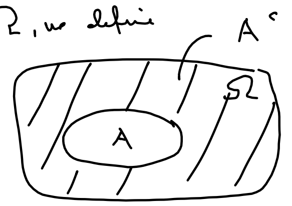
$$A \setminus B = \{a \in A : a \notin B\} = A \cap B^c$$





Complement: Given a universe Ω, we define

$$A^c = \{x \in \Omega : x \notin A\}$$



Distributive laws:

(i) $A \cap (B_1 \cup \dots \cup B_n) = (A \cap B_1) \cup \dots \cup (A \cap B_n)$

$A \cap \left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} (A \cap B_n)$

$a(b_1 + \dots + b_n) = ab_1 + \dots + ab_n$

(ii) $A \cup (B_1 \cap \dots \cap B_n) = (A \cup B_1) \cap \dots \cap (A \cup B_n)$

$A \cup \left(\bigcap_{n \in \mathbb{N}} B_n\right) = \bigcap_{n \in \mathbb{N}} (A \cup B_n)$

De Morgan's Laws:

(i) $(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$

$\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c = \bigcap_{n \in \mathbb{N}} A_n^c$

(ii) $(A_1 \cap A_2 \cap \dots \cap A_n)^c = A_1^c \cup A_2^c \cup \dots \cup A_n^c$

$\left(\bigcap_{n \in \mathbb{N}} A_n\right)^c = \bigcup_{n \in \mathbb{N}} (A_n^c)$

Proof: Assume $x \in \Omega$:

$x \in \left(\bigcap_{n \in \mathbb{N}} A_n\right)^c \Leftrightarrow x \notin \bigcap_{n \in \mathbb{N}} A_n \Leftrightarrow$ there is an n s.t. $x \notin A_n$

$\Leftrightarrow$ there is an n s.t. $x \in A_n^c \Leftrightarrow x \in \bigcup_{n \in \mathbb{N}} A_n^c$)

Probability Spaces

Fundamental question: What is probability?

Not going to answer!

Instead we shall make a mathematical framework that any coherent theory of prob. should satisfy.

Sample space: Ω = the sample space = the collection of all possible outcomes.

Examples: (i) Flip one coin: $\Omega = \{H, T\}$

(ii) Flip two coins: $\Omega = \{HH, HT, TH, TT\}$

(iii) Throw a die: $\Omega = \{1, 2, .., 6\}$

two dice: $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$

(iv) Select a number in $[0,1]$ at random:

$$\Omega = [0,1]$$

An event is a subset of the sample space:

Example: Two coins: Event: Get at least one H.

$$A = \{HH, HT, TH\} \subseteq \Omega.$$

Warning: If the sample space is infinite, there are subsets of Ω that are so complicated that we cannot give them a probability.

σ-fields and σ-algebras

"Idea: single out the subsets of Ω that we can assign a probability to."

Definition: Assume that Ω is a nonempty set. A σ-algebra / σ-field is a collection $\mathcal{F}$ of subsets of Ω such that:

$A_1 \cup A_2 \cup \dots \cup A_n \cup \emptyset \cup \dots \cup \emptyset$.

(i) $\emptyset \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(iii) If $A_n \in \mathcal{F}$ for all n, then $\underline{\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}}$.

Examples: (i) $\mathcal{F} = \{\emptyset, \Omega\}$

(ii) $\mathcal{F} = \mathcal{P}(\Omega)$ = all subsets of Ω.

(iii) If $\mathcal{A}$ is any collection of subsets of Ω, then there is a smallest σ-algebra $\sigma(\mathcal{A})$ that contains. This σ-algebra is called the σ-algebra generated by $\mathcal{A}$.

(iv) $\Omega = \mathbb{R}$, then the Borel σ-algebra on $\mathbb{R}$ is the smallest σ-algebra containing all open intervals (a, b).

Proposition: Assume that $\mathcal{F}$ is a σ-algebra on Ω. Then

a) $\Omega \in \mathcal{F}$

b) If $A_n \in \mathcal{F}$ for all n, then $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

c) If $A, B \in \mathcal{F}$, then $A \setminus B \in \mathcal{F}$

σ-algebra
(i) $\emptyset \in \mathcal{F}$
(ii) $A \in \mathcal{F} \Downarrow A^c \in \mathcal{F}$
(iii) $A_n \in \mathcal{F} \Downarrow \bigcup A_n \in \mathcal{F}$

Proof (a) $\emptyset \in \mathcal{F} \overset{ii}{\Rightarrow} \Omega = \emptyset^c \in \mathcal{F}$

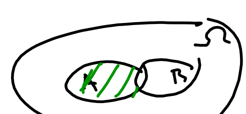
b) Since $A_n \in \mathcal{F} \Rightarrow A_n^c \in \mathcal{F}$ for all n

$$\Rightarrow \bigcup_{n\in\mathbb{N}} A_n^c \in \mathcal{F} \Rightarrow \Big(\bigcup_{n\in\mathbb{N}} A_n^c\Big)^c \in \mathcal{F}$$

$$\Big(\bigcup_{n\in\mathbb{N}} A_n^c\Big)^c = \bigcap_{n\in\mathbb{N}} (A_n^c)^c = \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{F}$$

c) Assume $A, B \in \mathcal{F}$, then

$$A \setminus B = \underset{\mathcal{F}}{\underset{\Uparrow}{A}} \cap \underset{\mathcal{F}}{\underset{\Uparrow}{B^c}} \in \mathcal{F}.$$



Probability measures

Assume that $\mathcal{F}$ is a σ-algebra on Ω. A probability measure P on $(\Omega, \mathcal{F})$ is a function

$$P : \mathcal{F} \to [0,1]$$

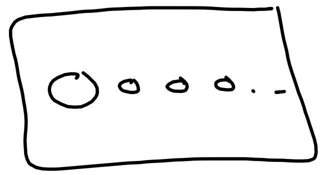
such that:

(i) $P(\emptyset) = 0$, $P(\Omega) = 1$

(ii) If $\{A_n\}$ is a disjoint sequence of sets in $\mathcal{F}$, then

$$P\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} P(A_n)$$

Def: The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.



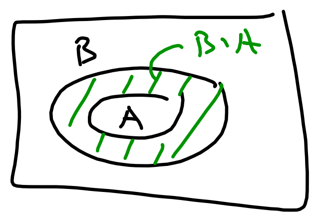
Proposition: $A, B \in \mathcal{F}$.

(i) If $A \subseteq B$, then $P(A) \leq P(B)$.

(ii) If $A \subseteq B$, then $P(B \setminus A) = P(B) - P(A)$. In particular, $P(A^c) = 1 - P(A)$.

(iii) If $A_n \in \mathcal{F}$ for all n, then

$$P\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \leq \sum_{n \in \mathbb{N}} P(A_n)$$



Proof: (i) + (ii). Assume $A \subseteq B$, then B is the disjoint union of A and $B \setminus A$: Hence

$$P(B) = P(A) + P(B \setminus A) \begin{cases} \nearrow P(B) \geq P(A) & \text{(i)} \\ \searrow P(B \setminus A) = P(B) - P(A) & \text{(ii)} \end{cases}$$

Put $B = \Omega$: $P(A^c) = P(\Omega \setminus A) = P(\Omega) - P(A) = 1 - P(A)$

(iii): Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus (A_1 \cup A_2)$

The B_n are disjoint and

$\bigcup B_n = \bigcup A_n$ Hence $\underset{A_n}{\overset{}{B_n \subseteq}}$

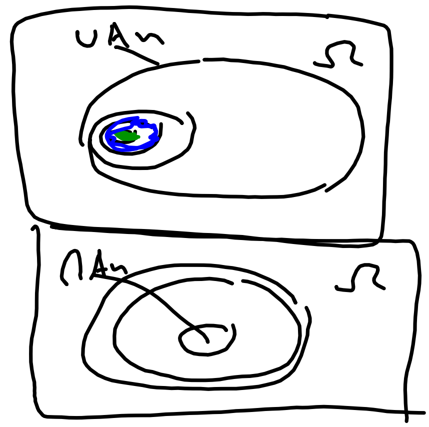
$$P\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = P\Big(\bigcup_{n \in \mathbb{N}} B_n\Big) = \sum_{n \in \mathbb{N}} P(B_n) \leq \sum_{n \in \mathbb{N}} P(A_n).$$

<u>Theorem</u> (Continuity of measure):

$A_n \subseteq A_{n+1}$

(i) Assume that $\{A_n\}$ is an $\underbrace{\text{increasing}}_{}$ sequence of elements in $\mathcal{F}$. Then

$$P\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} P(A_n)$$



(ii) Assume that $\{A_n\}$ is a decreasing sequence of element in $\mathcal{F}$. Then

$$P\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} P(A_n)$$

<u>Proof</u>: Let $B_0 = A_0$, $B_1 = A_1 \setminus A_0$, $B_2 = A_2 \setminus A_1$, .. Then

$$P\left(\bigcup_{n\in\mathbb{N}} A_n\right) = P\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n=1}^{\infty} P(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} P(B_n)$$

$$= \lim_{N\to\infty} P\left(\bigcup_{n=1}^{N} B_n\right) = \lim_{N\to\infty} P(A_N).$$