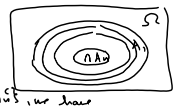
Continuity of measures: Measure space $(\Omega, \mathcal{F}, P)$.

(i) If $\{A_n\}$ is an increasing sequence of sets $A_n \in \mathcal{F}$, then $P(\bigcup_{n\in\mathbb{N}} A_n) = \lim P(A_n)$.

(ii) If $\{A_n\}$ is a decreasing sequence of sets $A_n \in \mathcal{F}$, then $P(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} P(A_n)$.



Proof of (ii): Note that $\{A_n^c\}$ is an increasing sequence with:

$\bigcup_{n\in\mathbb{N}} A_n^c = (\bigcap_{n\in\mathbb{N}} A_n)^c$. Applying (i) to $\{A_n^c\}$, we have

$$1-P(\bigcap_{n\in\mathbb{N}} A_n) = P(\bigcap A_n)^c = P(\bigcup A_n^c) = \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty}(1-P(A_n)) = 1-\lim_{n\to\infty} P(A_n)$$

Hence $P(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} P(A_n)$.

Notation: Main sets Ω, $\mathbb{R}$

Elements in Ω and $\mathbb{R}$ are (usually) denoted by l.c. letters $\omega, a, x, \ldots$

Subsets of Ω and $\mathbb{R}$ are (usually) by capital letters, $A, B, \ldots$

Families of subsets are usually denoted by script letters $\mathcal{F}, \mathcal{A}, \mathcal{G}$.

Construction of σ-algebras

Theorem: If $\mathcal{A}$ is any collection of subsets of Ω, then there is a smallest σ-algebra $\mathcal{F}$ such that $\mathcal{A} \subseteq \mathcal{F}$. (in the sense that if $\mathcal{G}$ is another σ-algebra with $\mathcal{A} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$)

Proof: Note that there is at least one σ-algebra containing $\mathcal{A}$, namely $\mathcal{P}(\Omega)$. Define

$\mathcal{F} = \{A \subseteq \Omega : A$ is an element of all σ-algebras $\mathcal{G}$ such that $\mathcal{A} \subseteq \mathcal{G}\}$

Obviously, $\mathcal{A} \subseteq \mathcal{F}$. Let us prove that $\mathcal{F}$ is a σ-algebra.

(i) $\emptyset \in \mathcal{F}$ since $\emptyset \in \mathcal{G}$ for all σ-algebras $\mathcal{G}$.

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$: If $A \in \mathcal{F}$, then $A \in \mathcal{G}$ for all $\mathcal{G}$, hence $A^c \in \mathcal{G}$ for all $\mathcal{G}$, and consequently $A^c \in \mathcal{F}$.

(iii) $A_n \in \mathcal{F}$ for all $n \Rightarrow \bigcup A_n \in \mathcal{F}$. Since $A_n \in \mathcal{F}$, then $A_n \in \mathcal{G}$ for all the $\mathcal{G}$'s and all the n's, hence $\bigcup A_n \in \mathcal{G}$ for all $\mathcal{G}$, and consequently $\bigcup A_n \in \mathcal{F}$.

Finally, note by construction $\mathcal{F} \subseteq \mathcal{G}$ for all $\mathcal{G} \supseteq \mathcal{A}$.

Definition: The smallest σ-algebra $\mathcal{F}$ containing $\mathcal{A}$ is called the σ-algebra generated by $\mathcal{A}$ and is denoted by $\sigma(\mathcal{A})$

Example: Let $\mathcal{A} = \{(a,b) : a, b \in \mathbb{R}, a < b\}$

$\sigma(\mathcal{A})$ = the Borel σ-algebra on $\mathbb{R}$

The ... $\sigma(\mathcal{A})$ are called Borel sets

Other kinds of families

Definition: Assume that Ω is a nonempty set. A collection $\mathcal{A}$ of subsets of Ω is called an algebra or a field if:

(i) $\emptyset \in \mathcal{A}$

(ii) If $B \in \mathcal{A}$, then $B^c \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.

Consequences:

(iv) $\Omega \in \mathcal{A}$

(v) If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{A}$ $\quad (A_1 \cup A_2 \cup A_3 = \underbrace{(A_1 \cup A_2)}_{\mathcal{A}} \cup A_3 \in \mathcal{A}$

(vi) If — " —, then $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}$ (De Morgan)

(vii) If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$ $\quad (A \setminus B = A \cap B^c)$

Example: Ω is a infinite set:

$\mathcal{A} = \{A \subseteq \Omega : A$ or A^c is finite$\}$.

is an algebra, but not a σ-algebra.

Definition: a collection $\mathcal{M}$ of subsets of Ω is called a monotone class if

(i) If $\{A_n\}$ is an increasing sequence of sets in $\mathcal{M}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$

(ii) If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{M}$, then $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{M}$

Observation: All σ-algebras are monotone classes. But there are monotone classes that are not σ-algebras

Example: Choose $\omega \in \Omega$, and let

$\mathcal{M} = \{A \subseteq \Omega : \omega \in A\}$

$\mathcal{M}$ is a monotone class that is not a σ-algebra

Monotone Class Theorem

Lemma: A monotone class $\mathcal{M}$ that is also an algebra, is a σ-algebra.

Proof: We need to check the conditions of a σ-algebra

(i) $\emptyset \in \mathcal{M}$ since $\mathcal{M}$ is an algebra

(ii) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ since $\mathcal{M}$ is an algebra

(iii) If $A_n \in \mathcal{M}$ for all n, we have to prove that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Set $B_1 = A_1, B_2 = A_1 \cup A_2, \ldots, B_n = A_1 \cup A_2 \cup \ldots \cup A_n$.

Since $\mathcal{M}$ is an algebra, all the B_n are in $\mathcal{M}$ and since $\{B_n\}$ is increasing

$\bigcup B_n \in \mathcal{M}$

since $\mathcal{M}$ is a monotone class. But

$\bigcup A_n = \bigcup B_n \in \mathcal{M}$.

Theorem: If $\mathcal{A}$ is a collection of subsets of Ω, then there is a smallest monotone class $\mathcal{M}(\mathcal{A})$ containing $\mathcal{A}$. It is called the monotone class generated by $\mathcal{A}$.

Proof: For you.

Monotone Class Theorem: If $\mathcal{A}$ is an algebra, then $\sigma(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

Proof: Note that $\underbrace{\mathcal{M}(\mathcal{A})}_{\text{the smallest} \ldots} \subseteq \sigma(\mathcal{A})$ since $\sigma(\mathcal{A})$ is a monotone class. [a monotone class containing $\mathcal{A}$] Assume that we can prove that $\mathcal{M}(\mathcal{A})$ is a σ-algebra, then

$\underbrace{\sigma(\mathcal{A})}_{\text{the smallest} \ldots \text{σ-algebra containing } \mathcal{A}} \subseteq \mathcal{M}(\mathcal{A})$

Hence it suffices to prove that $\mathcal{M}(\mathcal{A})$ is σ-algebra, and by the lemma it is actually enough to prove that $\mathcal{M}(\mathcal{A})$ is an algebra. Must check three axioms:

(i) $\emptyset \in \mathcal{A} \subseteq \mathcal{M}(\mathcal{A})$, thus $\emptyset \in \mathcal{M}(\mathcal{A})$.

(ii) Must prove that if $B \in \mathcal{M}(\mathcal{A})$, then $B^c \in \mathcal{M}(\mathcal{A})$.

Define: $\mathcal{C}_1 = \{B \in \mathcal{M}(\mathcal{A}) : B^c \in \mathcal{M}(\mathcal{A})\}$

Idea: prove that $\mathcal{C}_1$ is a monotone class containing $\mathcal{A}$. Then $\mathcal{C}_1 \supseteq \mathcal{M}(\mathcal{A})$ since $\mathcal{M}(\mathcal{A})$ is the smallest such monotone class, and thus any $B \in \mathcal{M}(\mathcal{A})$ would have $B^c \in \mathcal{M}(\mathcal{A})$.

1 Check that $\mathcal{C}_1$ is closed under increasing unions. $\{B_n\}$ is increasing and in $\mathcal{C}_1$.

Clearly $\bigcup B_n \in \mathcal{M}(\mathcal{A})$ and $(\bigcup B_n)^c = \bigcap_{n\in\mathbb{N}} B_n^c \in \mathcal{M}(\mathcal{A})$

2 $\mathcal{C}_1$ closed under decreasing intersections: $\{B_n\}$ decreasing and in $\mathcal{C}_1$

$\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{M}(\mathcal{A})$ and $(\bigcap_{n\in\mathbb{N}} B_n)^c = \bigcup_{n\in\mathbb{N}} B_n^c \in \mathcal{M}(\mathcal{A})$

Hence $\bigcap B_n \in \mathcal{C}_1$.

(iii) $A, B \in \mathcal{M}(\mathcal{A}) \Rightarrow A \cup B \in \mathcal{M}(\mathcal{A})$

Define: $\mathcal{C}_2 = \{B \in \mathcal{M}(\mathcal{A}) : B \cup A \in \mathcal{M}(\mathcal{A})$ for all $A \in \mathcal{A}\}$

Idea. Prove that $\mathcal{C}_2$ is a monotone class containing $\mathcal{A}$. This would mean that $\mathcal{C}_2 = \mathcal{M}(\mathcal{A})$ and hence that $B \cup A \in \mathcal{M}(\mathcal{A})$ for all $B \in \mathcal{M}(\mathcal{A})$ and all $A \in \mathcal{A}$.

Check that $\mathcal{C}_2$ is a monotone class:

Unions: Assume that $\{B_n\}$ is an increasing sequence from $\mathcal{C}_2$. Must prove $\bigcup B_n \in \mathcal{C}_2$. For all $A \in \mathcal{A}$:

$(\bigcup B_n) \cup A = \bigcup_{n\in\mathbb{N}} (B_n \cup A) \in \mathcal{M}(\mathcal{A})$ hence $\bigcup B_n \in \mathcal{C}_2$

Intersections: Assume that $\{B_n\}$ is a decreasing sequence from $\mathcal{C}_2$. Must prove $\bigcap B_n \in \mathcal{C}_2$: For all $A \in \mathcal{A}$

$(\bigcap_{n\in\mathbb{N}} B_n) \cup A \overset{\text{Distr.}}{=} \bigcap_{n\in\mathbb{N}} \underbrace{(B_n \cup A)}_{\mathcal{M}(\mathcal{A})} \in \mathcal{M}(\mathcal{A})$

Hence $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{C}_2$.

This means that $\mathcal{C}_2$ is a monotone class.

So we know that $B \cup A \in \mathcal{M}(\mathcal{A})$ when $B \in \mathcal{M}(\mathcal{A})$, $A \in \mathcal{A}$. Need to extend this to $A \in \mathcal{M}(\mathcal{A})$.

Define: $\mathcal{C}_3 = \{A \in \mathcal{M}(\mathcal{A}) : A \cup B \in \mathcal{M}(\mathcal{A})$ for all $B \in \mathcal{M}(\mathcal{A})\}$

Want to prove that $\mathcal{C}_3$ is a monotone class containing $\mathcal{A}$. If so $\mathcal{C}_3 = \mathcal{M}(\mathcal{A})$, and hence $A \cup B \in \mathcal{M}(\mathcal{A})$ for all $A, B \in \mathcal{M}(\mathcal{A})$.

By the previous step $\mathcal{A} \subseteq \mathcal{C}_3$. To prove that $\mathcal{C}_3$ is a monotone class, just repeat the arguments for $\mathcal{C}_2$.

This means that $A \cup B \in \mathcal{M}(\mathcal{A})$ for all $A, B \in \mathcal{M}(\mathcal{A})$ and hence $\mathcal{M}(\mathcal{A})$ is an algebra.

Application:

Theorem: Assume that $\mathcal{A}$ is an algebra and that P, Q are two prob measures on $\sigma(\mathcal{A})$. Assume that $P(A) = Q(A)$ for all $A \in \mathcal{A}$. Then $P(B) = Q(B)$ for all $B \in \sigma(\mathcal{A})$.

Idea of proof:

$\mathcal{M} = \{B \in \sigma(\mathcal{A}) : P(B) = Q(B)\}$

Prove that $\mathcal{M}$ is a monotone class.

Hence $P(B) = Q(B)$ for $B \in \mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$