Distribution functions

Definition: Assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $X: \Omega \to \mathbb{R}$. The distribution $\overset{\text{function}}{F}$ of X is defined by

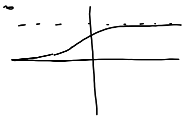
$$F(x) = P\{\omega : X(\omega) \leq x\}.$$

Property 1: F is an increasing function: Assume $y < x$

$$F(y) = P\{\omega : X(\omega) \leq y\} \leq P\{\omega : X(\omega) \leq x\} = F(x).$$

Property 2: $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$

Since F is increasing, it suffices to show that $\lim_{n \to \infty} F(n) = 1$



$$\lim_{n \to \infty} F(n) = \lim_{n \to \infty} P\{\omega : X(\omega) \leq n\}$$

$$\overset{\text{cont. measure}}{=} P\Big(\bigcup_{n \in \mathbb{N}} \{\omega : X(\omega) \leq n\}\Big) = P\{X(\omega) \in \mathbb{R}\} = 1.$$

To prove the second part, it suffices to prove that $\lim_{n \to -\infty} F(n) = 0$.

$$\lim_{n \to -\infty} F(n) = \lim_{n \to -\infty} P\{\omega : X(\omega) \leq n\} \overset{\text{cont. meas.}}{=} P\Big(\overbrace{\bigcap_{n \in -\mathbb{N}} \{\omega : X(\omega) \leq n\}}^{\emptyset}\Big)$$

$$P(\emptyset) = 0.$$

Property 3: F is right continuous, i.e. $\lim_{y \downarrow x} F(y) = F(x)$

Since F is increasing, it suffices to prove that

$$\lim_{n \to \infty} F(x + \tfrac{1}{n}) = F(x)$$

Proof

$$\lim_{n \to \infty} F(x + \tfrac{1}{n}) = \lim_{n \to \infty} P\{\omega : X(\omega) \leq x + \tfrac{1}{n}\} \overset{\text{cont. meas.}}{=} P\Big(\overbrace{\bigcap_{n \in \mathbb{N}} \{\omega : X(\omega) \leq x + \tfrac{1}{n}\}}^{\{\omega : X(\omega) \leq x\}}\Big)$$

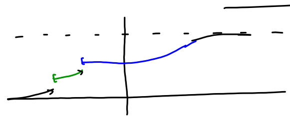
$$= F(x).$$

Property: $\lim_{y \uparrow x} F(y) = P\{\omega : X(\omega) < x\} \leq F(x)$

Proof: $\lim_{y \uparrow x} F(y) = \lim_{n \to \infty} F(x - \tfrac{1}{n}) = \lim_{n \to \infty} P\{\omega : X(\omega) < x - \tfrac{1}{n}\}$

$$= P\Big(\bigcup_{n \in \mathbb{N}} \{\omega : X(\omega) < x - \tfrac{1}{n}\}\Big)$$

$$= \underline{P\{\omega : X(\omega) < x\}} = F(x-)$$



Note that F is discontinuous at x if $F(x-) < F(x)$

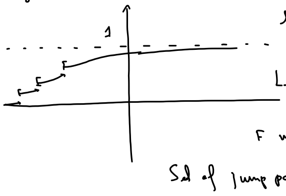
$$F(x) - F(x-) = P\{\omega : X(\omega) \leq x\} - P\{\omega : X(\omega) < x\}$$

$$= P\{\omega : X(\omega) = x\}$$

F is discontinuous at x iff $P\{\omega : X(\omega) = x\} > 0$.

Property 5: F has at most countably many points of discontinuity.

Proof: How many jumps can there be of size larger than $\frac{1}{n}$. $\left(F(x) - F(x-) \geq \frac{1}{n} \right)$ We can have at most n such gaps.

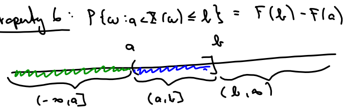


Let A_n be the collection of x's where F makes a jump of size $\geq \frac{1}{n}$

Set of jump points $= \bigcup_{n \in \mathbb{N}} A_n$ "=" "countable union of finite sets" = "countable"

$$P\{\omega : X(\omega) \in (a, b]\}$$
$$\text{"}$$

Property 6: $P\{\omega : a < X(\omega) \leq b\} = F(b) - F(a)$



$$F(b) = P\{\omega : \underline{X(\omega) \leq b}\} = P\{\omega : \underbrace{\underline{X(\omega) \leq a}}_{F(a)}\} + P\{\omega : \underline{a < X(\omega) \leq b}\}$$

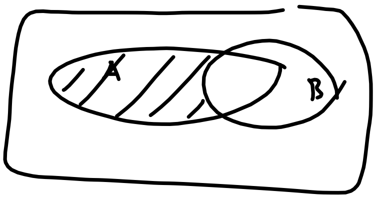
$$\underset{F(b)}{\text{"}}$$

Hence $P\{\omega : a < X(\omega) \leq b\} = F(b) - F(a)$

Property 7: $P\{\omega : X(\omega) > x\} = 1 - F(x)$

1.5, 1.6, 1.8, 1.9, 1.16 have solutions from last years.

Problem 1.3: $A, B, C \subseteq \Omega$



$A - B = A \cap B^c$

a) $(A-B)-C = A \cap B^c \cap C^c$

Proof: $(A-B)-C = (A-B) \cap C^c$
$= (A \cap B^c) \cap C^c = A \cap B^c \cap C^c$

b) $(A \cap B) - C = (A-C) \cap (B-C)$

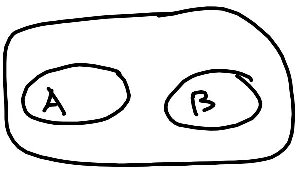
Proof: $(A \cap B) - C = (A \cap B) \cap C^c = (A \cap C^c) \cap (B \cap C^c)$
$= (A - C) \cap (B - C)$

c) $A \cup B = B \iff A \subseteq B$ $\subset$.

$\Rightarrow$ Assume that $A \cup B = B$. If $x \in A$, then $x \in A \cup B = B$, hence $A \subseteq B$.

$\Leftarrow$ Assume that $A \subseteq B$. Then $A \cup B = B$

Problem 1.21 Ω, A, B, $P(A) = 0.4$, $P(B) = 0.5$
$A \cap B = \emptyset$



$P(A \cup B) = P(A) + P(B) = 0.9$

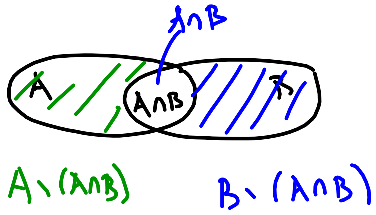
$P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$

$P(A^c \cap B) = P(B) = 0.5.$

since $B \subseteq A^c$

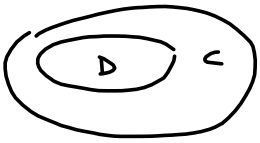
Problem 1.22: $P(A \cup B) + P(A \cap B) = P(A) + P(B)$

more commonly written $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$$P(A \cup B) = P(A \setminus (A \cap B)) + P(A \cap B) + P(B \setminus (A \cap B))$$

$$= P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(A \cap B)$$



$$P(C \setminus D) = P(C) - P(D)$$

$$= P(A) + P(B) - P(A \cap B)$$

Problem 1.23: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Solution: $P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(\underline{(A \cap C) \cup (B \cap C)})$$

$$= P(A) + P(B) + P(C) - \underline{P(A \cap B)} - [\underline{P(A \cap C)} + \underline{P(B \cap C)} - P(\underbrace{(A \cap C) \cap (B \cap C)})]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Rule of inclusion and exclusion.

Problem 1.5: Show that the following are algebras (fields) and not σ-algebras (σ-fields)

a) All finite subsets and their complements

b) $\emptyset$ and all intervals $(-\infty, a]$, $(b, c]$, (d, ∞)

Need to check:

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

} algebra

(iii') $\{A_n\}$ a sequence in $\mathcal{A} \Rightarrow \bigcup A_n \in \mathcal{A}$.

a) $\mathcal{A} = \{A : A \text{ is finite or } A^c \text{ is finite}\}$

Check the properties:

(i) $\emptyset$ is finite $\Rightarrow \emptyset \in \mathcal{A}$.

(ii) Assume that $A \in \mathcal{A}$. Two possibilities

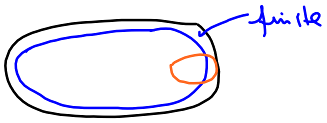
A finite $\Rightarrow (A^c)^c = A$ (finite), so $A^c \in \mathcal{A}$ because it has a finite compl.

A^c finite $\Rightarrow A^c \in \mathcal{A}$.

(iii) Assume $A, B \in \mathcal{A}$. Two situations:

Both A, B finite $\Rightarrow A \cup B$ finite $\Rightarrow A \cup B \in \mathcal{A}$.

At least one of A, B has a finite complement.



Since $A \cup B$ is larger than both of them, it must also have a finite complement, and hence $A \cup B \in \mathcal{A}$.

Why isn't $\mathcal{A}$ a σ-algebra (when Ω is infinite)

Make a list of different elements.

$\omega_1, \omega_2, \omega_3, \omega_4, \ldots\ldots$

$A_1 = \{\omega_1\}, A_2 = \{\omega_3\}, A_3 = \{\omega_5\}, \ldots \in \mathcal{A}$

$\bigcup_{n \in \mathbb{N}} A_n = \{\omega_1, \omega_3, \omega_5 \ldots\}$ — not finite, not finite complement

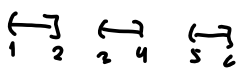
$\bigcup A_n \notin \mathcal{A}$.

b) $\underbrace{\text{All left-open, right-closed intervals}}_{\text{finite unions of}}$, $\underline{\emptyset}$

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.

(iii) Unions (trivial!)



— not in $\mathcal{A}$.