STK-MAT3710: The Fourier Inversion Theorem

The textbook defines the Fourier transform of a random variable X as

$$\phi(t) = E\left[e^{itX}\right]$$

If X has a density f, this becomes

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

The Fourier Inversion Theorem says that under suitable conditions, we can for almost all x recover the original density f from its Fourier transform ϕ by

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \phi(t) e^{-itx}\, dt \tag{1}$$

It would take us too far afield to prove this theorem in the present course, but I would like to give you some indication of why it is true.

If we write ϕ as

$$\phi(t) = \int_{-\infty}^{\infty} e^{ity} f(y)\, dy$$

(we need the variable x for other purposes), we get

$$\frac{1}{2\pi}\int_{-T}^{T} \phi(t) e^{itx}\, dt = \frac{1}{2\pi}\int_{-T}^{T}\left(\int_{-\infty}^{\infty} e^{ity} f(y)\, dy\right) e^{-itx}\, dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\left(\int_{-\infty}^{\infty} e^{it(y-x)} f(y)\right) dy\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left(\int_{-T}^{T} e^{it(y-x)}\, dt\right) dy$$

where we in the last step have changed the order of integration (as this is just an informal calculation, we are not very careful with conditions). Integrating the inner integral, we get

$$\int_{-T}^{T} e^{it(y-x)}\, dt = \frac{e^{iT(y-x)} - e^{-iT(y-x)}}{i(y-x)}$$

$$= \frac{\big(\cos[T(y-x)] + i\sin[T(y-x)]\big) - \big(\cos[-T(y-x)] + i\sin[-T(y-x)]\big)}{i(y-x)}$$

$$= \frac{2\sin[T(y-x)]}{y-x}\ ,$$

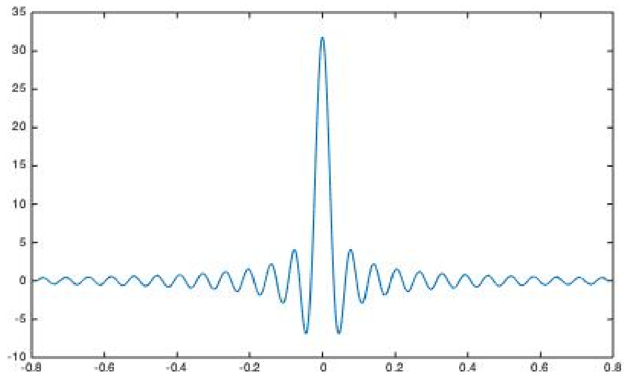
and hence we have

$$\frac{1}{2\pi}\int_{-T}^{T} \phi(t) e^{itx}\, dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{\sin[T(y-x)]}{y-x}\, dy = \int_{-\infty}^{\infty} f(y) g_T(y-x)\, dy\ ,$$

where g_T is the integral kernel

$$g_T(u) = \frac{\sin(Tu)}{\pi u}\ .$$

The figure below shows g_T for $T = 100$, and it is typical for what g_T looks like for large values of T: There is a sharp peak as the origin and smaller waves running off to $\pm\infty$.



As T increases, the peak gets higher and higher and narrower and narrower, but some properties are conserved: For instance, if we introduce a new variable $z = Tu$, we get $du = \frac{dz}{T}$, and hence

$$\int_{-\infty}^{\infty} g_T(u)\, du = \int_{-\infty}^{\infty} \frac{\sin(Tu)}{\pi u}\, du = \int_{-\infty}^{\infty} \frac{\sin z}{\pi z}\, dz = 1$$

(we are using that $\int_{-\infty}^{\infty} \frac{\sin u}{u}\, du = \pi$ — this is not an elementary integral, but can be computed with techniques from complex analysis). This means that the integral

$$\int_{-\infty}^{\infty} f(y) g_T(y - x)\, dy$$

can be thought of as a weighted average of the values of f. Since g_T has a peak at the origin, the values $f(y)$ where y is close to x get the highest weights, and as the peak gets higher and higher and narrower and narrower, this effect becomes increasingly stronger as T increases. Hence it is not surprising that

$$f(x) = \lim_{T\to\infty} \int_{-\infty}^{\infty} f(y) g_T(y - x)\, dy\ ,$$

at least at points x where f is continuous (if f has a jump discontinuity at x, we will get $\frac{1}{2}\big[f(x_+) + f(x_-)\big]$ instead since the kernel g_T is symmetric.) As

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} f(y) g_T(y - x)\, dy = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \phi(t) e^{-itx}\, dt\ ,$$

we have now completed our heuristic argument for the inversion formula (1).

Formal proofs of the Fourier Inversion Theorem can be found in a number of books, e.g. T. Körner: *Fourier Analysis*, H.L. Montgomery: *Early Fourier Analysis*, and P. Billingsley: *Probability and Measure*. Be aware that there is no ultimate version of the Fourier Inversion Theorem, and that different books will present slightly different versions. Also be aware that some books define the Fourier transform by

$$\phi(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i t x} \, dx$$

(note the 2π in the exponent). The inversion formula then becomes

$$f(x) = \lim_{T \to \infty} \int_{-T}^{T} \phi(t) e^{-2\pi i t x} \, dt \tag{2}$$