STK-MAT3710: Trial Exam 2. Fall 2019

Problem 1. Let X_1, X_2 and Y_1, Y_2 be random variables. Assume that X_1 and Y_1 have the same distribution, and so do X_2 og Y_2.

a) Show that if X_1 and X_2 are independent, and the same is the case for Y_1 and Y_2, then $X_1 + X_2$ and $Y_1 + Y_2$ have the same distribution.

b) Show by an example that if we drop the condition that the variables are independent, then X_1+X_2 and Y_1+Y_2 need not have the same distribution.

Problem 2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $H : \Omega \to \Omega$ is called *measure preserving* if the following two conditions are satisfied:

(i) If $A \in \mathcal{F}$, then $H^{-1}(A) \in \mathcal{F}$.

(ii) If $A \in \mathcal{F}$, then $P(H^{-1}(A)) = P(A)$.

If $Y : \Omega \to \mathbb{R}$ is a random variable, we define $Y_H : \Omega \to \mathbb{R}$ by $Y_H(\omega) = Y(H(\omega))$.

a) Show that Y_H is a random variable.

b) Show that if Y is a discrete, integrable rnadom variable, then $E(Y) = E(Y_H)$.

c) Show that if Y is an integrable random variable, then $E(Y) = E(Y_H)$.

Problem 3. Let $\mathcal{T} = \{0, 1, 2, \ldots\}$ be a timeline and $\{\mathcal{F}_n\}_{n\in\mathcal{T}}$ a filtration. Assume that $\{X_n\}_{n\in\mathcal{T}}$ is a bounded, adapted, process, and that $\{M_n\}_{n\in\mathcal{T}}$ is a $\mathcal{F}_n$-martingale. Let $\Delta M_k = M_{k+1} - M_k$ be the forward increment of X at time k.

a) Define a new process $\{Y_n\}$ by $Y_0 = 0$ and $Y_n = \sum_{k=0}^{n-1} X_k \Delta M_k$. Show that $\{Y_n\}$ is an $\mathcal{F}_n$-martingale.

b) Assume that M_k^2 is integrable for all k. Show that the variance of Y_n is $\sum_{k=0}^{n-1} E\left[X_k^2 \Delta M_k^2\right]$.

Problem 4.

a) Assume that $\{Y_n\}$ is a sequence of random variables converging in distribution to a random variable Y. Show that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(Y_n)$ converges in distribution to $g(Y)$.

In the rest of problem, $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of bounded, independent and identically distributed random variables with first moment $E[X_n] = 0$ and second moment $E[X_n^2] = \alpha > 0$. Note that the X_n's are bounded; i.e. there exists a real number M such that $|X_n| \leq M$.

b) La $P_n = \prod_{k=1}^{n}(1 + \frac{X_k}{\sqrt{n}})$. Show that $\log P_n$ converges in distribution to a normal distribution. You may use without proof that if U_n converges to U in distribution and V_n converges to a constant C in probability, then $U_n + V_n$ converges to $U + C$ in distribution.

c) Show that P_n converges in distribution an describe the limit.