STK-MAT3710: Trial Exam 2. Fall 2019. Solutions

Problem 1. As X_1 and Y_1 have the the same distribution, they have the same characteristic function ϕ_1, and for the same reason X_2 and Y_2 have the same characteristic function ϕ_2. By independence,

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t) = \phi_1(t)\phi_2(t)\,,$$

and similarly

$$\phi_{Y_1+Y_2}(t) = \phi_{Y_1}(t)\phi_{Y_2}(t) = \phi_1(t)\phi_2(t)\,.$$

As $X_1 + X_2$ and $Y_1 + Y_2$ have the same characteristic function, they also have the same distribution.

b) Let X_1 be a random variable that takes values 1 and -1 with probability $\frac{1}{2}$, and let $X_2 = X_1$. Let Y_1 and Y_2 be two *independent* copies of X_1. Then $X_1 + X_2$ takes the values 2 and -2 with probability $\frac{1}{2}$ while $Y_1 + Y_2$ takes three values 2, 0, and -2 with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively,

Problem 2. a) We have

$$\begin{aligned}\{\omega : Y_H(\omega) \leq x\} &= \{\omega : Y(H(\omega)) \leq x\} \\ &= H^{-1}(\{\omega : Y(\omega) \leq x\}) \in \mathcal{F}\end{aligned}$$

where we have used that Y is a random variable in combination with condition (i) in the problem.

b) Since $Y = \sum a_i \mathbf{1}_{A_i}$, we have $E[Y] = \sum a_i P(A_i)$. On the other hand,

$$Y_H(\omega) = Y(H(\omega)) = \sum a_i \mathbf{1}_{A_i}(H(\omega)) = \sum a_i \mathbf{1}_{H^{-1}(A_i)}(\omega)$$

Hence

$$E[Y_H] = \sum a_i P(H^{-1}(A_i)) = \sum a_i P(A_i) = E[Y]$$

where we have used property (ii) in the problem.

c) Let

$$\underline{Y}_n = \sum_{n\in\mathcal{Z}} \frac{k}{2^n} \mathbf{1}_{[k2^{-n} < Y \leq (k+1)2^{-n}]}$$

be the lower approximation to Y. Then

$$(\underline{Y}_n)_H = \sum_{n\in\mathcal{Z}} \frac{k}{2^n} \mathbf{1}_{H^{-1}([k2^{-n} < Y \leq (k+1)2^{-n}])}$$

is the corresponding lower approximation of Y_H since

$$H^{-1}([k2^{-n} < Y \leq (k+1)2^{-n}]) = [k2^{-n} < Y_H \leq (k+1)2^{-n}].$$

By b), $E[\underline{Y}_n] = E[(\underline{Y}_n)_H]$, and since by definition we have $E[\underline{Y}_n] \to E[Y]$ and $E[(\underline{Y}_n)_H] \to E[Y_H]$, we get $E[Y] = E[Y_H]$.

Problem 3. a) Let B be a bound for $|X_n|$. To see that Y is integrable, note that $|\Delta M_k| = |M_{k+1} - M_k| \le |M_{k+1}| + |M_k|$, and hence ΔM_k is integrable. As

$$|Y_n| = |\sum_{k=0}^{n-1} X_k \Delta M_k| \le \sum_{k=0}^{n-1} |X_k||\Delta M_k| \le B \sum_{k=0}^{n-1} |\Delta M_k|$$

we see that Y_n is integrable. As $\{Y_n\}$ is clearly adapted, it only remains to prove the martingale property. We have

$$E[Y_{n+1}|\mathcal{F}_n] = E[X_n \Delta M_n|\mathcal{F}_n] + E[Y_n|\mathcal{F}_n] = X_n E[\Delta M_n|\mathcal{F}_n] + Y_n = Y_n\ ,$$

where we have used that X_n and Y_n are $\mathcal{F}_n$-measurable, and that $E[\Delta M_n|\mathcal{F}_n] = 0$ as M is a martingale.

b) Since Y is a martingale $E[Y_n] = E[Y_0] = 0$, and hence

$$\mathrm{Var}(Y_n) = E[Y_n^2] = E\left[\left(\sum_{k=0}^{n-1} X_k \Delta M_k\right)^2\right]$$

As M is square integrable and X is bounded, the sum $\sum_{k=0}^{n-1} X_k \Delta M_k$ is square integrable, and hence the variance is finite. Multiplying out the parenthesis and regrouping the terms, we get:

$$\mathrm{Var}(Y_n) = E\left[\left(\sum_{k=0}^{n-1} X_k \Delta M_k\right)^2\right] = E\left[\sum_{i,j=1}^{n-1} X_i \Delta M_i X_j \Delta M_j\right]$$

$$= 2 \sum_{1 \le i < j \le n-1} E\big[X_i \Delta M_i X_j \Delta M_j\big] + \sum_{k=0}^{n-1} E\big[X_k^2 \Delta M_k^2\big]$$

In the product $X_i \Delta M_i X_j \Delta M_j$ all factors are $\mathcal{F}_j$-measurable except ΔM_j, and hence by the tower property of conditional expectations:

$$E\big[X_i \Delta M_i X_j \Delta M_j\big] = E\Big[E\big[X_i \Delta M_i X_j \Delta M_j|\mathcal{F}_j\big]\Big] = E\Big[X_i \Delta M_i X_j E\big[\Delta M_j|\mathcal{F}_j\big]\Big] = 0$$

as $E[\Delta M_j|\mathcal{F}_j] = 0$ since M is a martingale.

Problem 4. a) Since convergence in distribution is the same as weak convergence, it suffices to show that $E[f(g(Y_n))] \to E[(f(g(Y))]$ for all bounded, continuous functions. But if f is bounded and continuous, so is the composition $f \circ g(x) = f(g(x))$, and as Y_n converges weakly to Y, we have $E[f(g(Y_n))] \to E[(f(g(Y))]$.

b) Put $Y_n = \log P(n)$. Using the Taylor expansion $\log(1+x) = x - \frac{x^2}{2} + R(x)$, where $R(x) = \frac{x^3}{3(1+c)^3}$ for some c between 0 and x, we get

$$Y_n = \log\left(\prod_{k=1}^{n}\left(1+\frac{X_k}{\sqrt{n}}\right)\right) = \sum_{k=1}^{n}\log\left(1+\frac{X_k}{\sqrt{n}}\right) = \sum_{k=1}^{n}\left(\frac{X_k}{\sqrt{n}} - \frac{X_k^2}{2n} + R\left(\frac{X_k}{\sqrt{n}}\right)\right)$$

By the Central Limit Theorem, the first part of the sum, $\sum_{k=1}^{n}\frac{X_k}{\sqrt{n}}$, converges in distribution to a normal distribution with mean $E[X_n] = 0$ and variance $\mathrm{Var}(X_n) = \alpha > 0$. By the Weak Law of Large Numbers, the second part of the sum, $\sum_{k=1}^{n}\frac{X_k^2}{2n}$, converges in probability to $\frac{1}{2}E[X_k^2] = \frac{\alpha}{2}$. The third part of the sum, $\sum_{k=1}^{n} R(\frac{X_k}{\sqrt{n}})$, converges pointwise to zero as the following argument shows: As $|X_k|$ is bounded by M, we have $\frac{|X_k|}{\sqrt{n}} < \frac{1}{2}$ for sufficiently large n, and hence $|1+c| > \frac{1}{2}$ in the expression for the error term. For such n, we have

$$|R\left(\frac{X_k}{\sqrt{n}}\right)| \leq \frac{|X_k|^3}{3\cdot(\frac{1}{2})^3 n^{3/2}} \leq \frac{8M^3}{3n^{3/2}}$$

Hence $\sum_{k=1}^{n} R(\frac{X_k}{\sqrt{n}}) \to 0$ as we are summing n terms of size $\frac{1}{n^{3/2}}$.

This means that Y_n is the sum of a term converging in distribution a normal distribution $N(0,\alpha)$ and a term converging in probability to the constant $-\frac{\alpha}{2}$. By the result cited in the problem text, this means that Y_n converges in distribution to a $N(-\frac{\alpha}{2},\alpha)$ random variable.

c) Combining parts a) and b), we see that $P_n = e^{Y_n}$ converges in distribution to e^Z, where Z is a $N(-\frac{\alpha}{2},\alpha)$ random variable.