UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK-MAT3710/4710 — Probability Theory.
Day of examination:	Friday, December 11th, 2020.
Examination hours:	15.00 – 19.00.
This problem set consists of 2 pages.	
Appendices:	Formula sheet.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

We use $\mathbb{N}_0$ to denote the natural numbers with 0 included, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

Problem 1 (50 points)

In this problem λ is a positive real number.

a) Let Y be a Poisson random variable with intensity λ, i.e. Y is taking values in the nonnegative integers $\mathbb{N}_0$, and

$$P[Y = n] = \frac{\lambda^n}{n!} e^{-\lambda}$$

for each $n \in \mathbb{N}_0$. Show that the characteristic function of Y is

$$\phi_Y(t) = e^{\lambda(e^{it} - 1)}.$$

b) Find $E[Y]$, e.g. by differentiating ϕ_Y.

c) Let $n \in \mathbb{N}$ and assume that $n > \lambda$. Let X_n be a random variable taking only two values 0 and 1 with probabilities $P[X_n = 1] = \frac{\lambda}{n}$ and $P[X_n = 0] = 1 - \frac{\lambda}{n}$, respectively. Show that the characteristic function of X_n is

$$\phi_{X_n}(t) = 1 + \frac{\lambda}{n}(e^{it} - 1).$$

d) Let $S_n = X_n^{(1)} + X_n^{(2)} + \ldots + X_n^{(n)}$ be the sum of n independent copies of X_n. Find the characteristic function of S_n.

e) Show that the sequence $\{S_n\}_{n \in \mathbb{N}}$ converges to Y in distribution (here Y is the random variable in question a)).

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Problem 2 (20 points)

In this problem a and b are real numbers such that $b < 0 < a$. Assume that p is a real number between 0 and 1, and let X be a random variable taking the values a and b with probabilities $P[X = a] = p$, $P[X = b] = 1 - p$, respectively.

Assume that $\{X_j\}_{j\in\mathbb{N}}$ is an independent sequence of copies of X, and define a process $M = \{M_n\}_{n\in\mathbb{N}_0}$ by putting $M_0 = 0$ and $M_n = \sum_{j=1}^{n} X_j$ for $n > 0$. Define the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ by letting $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ for $n > 0$.

a) For which value of p is M a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$? For which values of p is it a submartingale and for which is it a supermartingale?

b) Assume from now on that $a = 1$ and $b = -1$. Pick two integers k, m such that $k < 0 < m$ and define

$$T(\omega) = \inf\{n \in \mathbb{N}_0 \colon M_n(\omega) = k \text{ or } M_n(\omega) = m\}.$$

Define $M_T(\omega) = M_{T(\omega)}(\omega)$ (you may assume without proof that T is finite a.s.). For which values of p is $E[M_T] > 0$ and for which values of p is $E[M_T] < 0$? Explain your reasoning.

Problem 3 (40 points)

In this problem you may use that a finite sum of gaussian random variables is gaussian.

a) Let $\{X_n\}_{n\in\mathbb{N}}$ be an independent sequence of gaussian random variables with mean zero and variance 1. Put $Y_N = X_1 + X_2 + \cdots + X_N$. Find the mean and variance of Y_N.

b) Explain that if $a > 0$, then

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \le \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\,dx$$

and use this to prove that

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \le \frac{1}{a} e^{-\frac{a^2}{2}}.$$

c) Let $\epsilon > 0$. Show that for all $N \in \mathbb{N}$,

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] \le \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi}}.$$

d) Prove that

$$P\left[Y_N > \sqrt{N^{1+\epsilon}} \text{ for infinitely many } N\right] = 0.$$

You may use without proof that $\lim_{N\to\infty} \frac{N^p}{e^{\frac{N^\epsilon}{2}}} = 0$ for all $p > 0$.

THE END