29th September, 2020

STK-MAT3710/4710

Mandatory assignment 1 of 1

Submission deadline

Thursday 15th October 2020, 14:30 in Canvas (canvas.uio.no).

Instructions

If you are taking STK-MAT3710, you can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). If you are taking STK-MAT4710, you have to write the assignment in LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. You need a 60 % score to pass this assignment. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

All items (1, 2a, 2b etc.) count equally, and you need a score of at least 60 % to pass. Note that if there is an item you cannot do, you may still use the result in later problems.

Problem 1. Assume that $\{A_n\}_{n\in\mathbb{N}}$ is an independent sequence of events. Show that

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} P(A_n),$$

where the infinite product is defined as $\prod_{n=1}^{\infty} P(A_n) = \lim_{N\to\infty} \prod_{n=1}^{N} P(A_n)$.

Problem 2. In this problem, $(X, \mathcal{A}, P)$ and $(Y, \mathcal{B}, Q)$ are two probability spaces. A set $R \subseteq X \times Y$ is called a *measurable rectangle* if $R = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

a) Show that if R, S are two measurable rectangles, then $R \cap S$ is a measurable rectangle.

b) Show that if R is a measurable rectangle, then $R^c = (X \times Y) \setminus R$ is the disjoint union of two measurable rectangles.

Let $\mathcal{R}$ be the collection of all subsets of $X \times Y$ which can be written as a finite, disjoint union of measurable rectangles (hence a subset R of $X \times Y$ is in $\mathcal{R}$ if there are disjoint measurable rectangles $R_1, R_2, \ldots, R_n$ such that $R = R_1 \cup R_2 \cup \ldots \cup R_n$). Our next goal is to show that $\mathcal{R}$ is the algebra generated by the measurable rectangles (i.e. the smallest algebra containing all measurable rectangles).

c) Show that if $R, S \in \mathcal{R}$, then $R \cap S \in \mathcal{R}$.

d) Extend c) to more than two sets, i.e. show that if $S_1, S_2, \ldots, S_n \in \mathcal{R}$, then $S_1 \cap S_2 \cap \ldots \cap S_n \in \mathcal{R}$ for all $n \in \mathbb{N}$.

e) Show that if $R \in \mathcal{R}$, then $R^c \in \mathcal{R}$.

f) Show that $\mathcal{R}$ is the algebra generated by the measurable rectangles.

Assume $E \subseteq X \times Y$. For each $x \in X$, define the *section* $E^x \subseteq Y$ by

$$E^x = \{y \in Y : (x, y) \in E\}$$

(it may be helpful to make a drawing).

g) Show that $(E^x)^c = (E^c)^x$ and that $\left(\bigcup_{n=1}^{\infty} E_n\right)^x = \bigcup_{n=1}^{\infty} (E_n)^x$.

h) Let $\mathcal{E}$ be the σ-algebra generated by the collection of all measurable rectangles and let $x \in X$. Show that if E is $\mathcal{E}$-measurable, then E^x is $\mathcal{B}$-measurable. (*Hint:* Let $\mathcal{D}$ be the collection of all sets in $\mathcal{E}$ such that $E^x \in \mathcal{B}$, and show that $\mathcal{D}$ is a σ-algebra.)

One can show that there is a probability measure $P \times Q$ on $\mathcal{E}$ such that

$$(P \times Q)(A \times B) = P(A)Q(B)$$

for all measurable rectangles $A \times B$ (we don't have the machinery to prove this in the present course, and we shall just take $P \times Q$ for granted).

i) Let $\mathcal{M}$ be the collection of all sets $E \in \mathcal{E}$ such that

   (i) the function $x \mapsto Q(E^x)$ is $\mathcal{A}$-measurable, and

   (ii) $(P \times Q)(E) = \int Q(E^x)\, dP(x)$.

   Show that $\mathcal{M}$ is a monotone class.

j) Explain that

$$(P \times Q)(E) = \int Q(E^x)\, dP(x)$$

   for all $E \in \mathcal{E}$.

k) Show that for all nonnegative, $\mathcal{E}$-measurable random variables $U\colon X \times Y \to \mathbb{R}$, we have

$$E_{P\times Q}(U) = E_P\left(\int U(x,y)\, dQ(y)\right)$$

   where $E_{P\times Q}$ denotes expectation with respect to $P \times Q$, E_P denotes expectation with respect to P, and $\int U(x,y)\, dQ(y)$ denotes the integral of the function $y \mapsto U(x,y)$ with respect to Q as x is kept fixed.

Good Luck!