STK-MAT3710/4710 Fall 2020: Solution to the mandatory assignmeent.

Problem 1. Let $B_N = \bigcap_{n=1}^N A_n$. Then $\{B_N\}$ is a decreasing sequence with $\bigcap_{n=1}^\infty B_n = \bigcap_{n=1}^\infty A_n$. By continuity of measure,

$$P\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{N\to\infty} P(B_N) = \lim_{N\to\infty} P\left(\bigcap_{n=1}^N A_n\right)$$

By independence, $P(\bigcap_{n=1}^N A_n) = \prod_{n=1}^N P(A_n)$, and hence

$$P\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{N\to\infty} P\left(\bigcap_{n=1}^N A_n\right) = \lim_{N\to\infty} \prod_{n=1}^N P(A_n) = \prod_{n=1}^\infty P(A_n).$$

Problem 2. a) Let $R = A\times B$, $S = B\times D$. Then $R\cap S = (A\cap C)\times(B\cap D)$, which is a measurable rectangle since $\mathcal{A}$ and $\mathcal{B}$ are closed under finite intersections.

b) Let $R = A\times B$. Then $(A\times B)^c = (A^c\times Y)\cup(A\times B^c)$, which is a disjoint union of two measurable rectangles.

c) Assume that $R = R_1\cup R_2\cup\ldots\cup R_n$ and $S = S_1\cup S_2\cup\ldots\cup S_m$ where the unions consist of disjoint measurable rectangles. Then

$$R\cap S = \bigcup_{i,j}(R_i\cap S_j)$$

where the elements of the union are disjoint, measurable rectangles.

d) We use induction on n: Assume that $S_1, S_2, \ldots, S_n$ are elements of $\mathcal{R}$. If $n = 1$ or $n = 2$, we know that the intersection $S_1\cap S_2\cap\ldots\cap S_n$ is in $\mathcal{R}$. To check the induction step, note that if the assumption holds for $n = k$, it also holds for $n = k+1$ as

$$S_1\cap S_2\cap\ldots\cap S_k\cap S_{k+1} = (S_1\cap S_2\cap\ldots\cap S_k)\cap S_{k+1}.$$

Here the first set is in $\mathcal{R}$ by the induction hypothesis and the second by assumption, and hence the intersection is in $\mathcal{R}$ by c). The assertion now follows by induction.

e) Assume that $R = R_1\cup R_2\cup\ldots\cup R_n$ where the union consists of disjoint measurable rectangles. By one of De Morgan's laws,

$$R^c = R_1^c\cap R_2^c\cap\ldots R_n^c.$$

By b), each R_i^c is the disjoint union of two measurable rectangles and hence in $\mathcal{R}$, and as we have just proved that $\mathcal{R}$ is closed under finite intersections, $R^c\in\mathcal{R}$.

f) As any algebra containing all measurable rectangles has to include all finite unions of measurable rectangles, it suffices to show that $\mathcal{R}$ is an algebra. There are three conditions to check:

(i) $\emptyset\in\mathcal{R}$

(ii) If $R \in \mathcal{R}$, then $R^c \in \mathcal{R}$.

(iii) If $R, S \in \mathcal{R}$, the $R \cup S \in \mathcal{R}$

To prove (i), just note that $\emptyset = \emptyset \times \emptyset \in \mathcal{R}$. As (ii) is identical to d), we only need to prove (iii). This is just a little exercise in De Morgan: As $R \cup S = (R^c \cap S^c)^c$, we see that $R \cup S \in \mathcal{R}$ by combining c) and d).

g) To prove the assertion about complements, just note that

$$y \in (E^x)^c \iff y \notin E^x \iff (x, y) \notin E \iff (x, y) \in E^c \iff y \in (E^c)^x$$

The proof for unions is similar:

$$y \in \left(\bigcup_{n=1}^{\infty} E_n \right)^x \iff (x, y) \in \bigcup_{n=1}^{\infty} E_n \iff (x, y) \in E_n \text{ for at least one } n$$

$$\iff y \in (E_n)^x \text{ for at least one } n \iff y \in \bigcup_{n=1}^{\infty} (E_n)^x$$

h) Let $\mathcal{D}$ be the collection of all sets in $\mathcal{E}$ such that $E^x \in \mathcal{B}$. It suffices to prove that $\mathcal{D}$ is a σ-algebra containing all measurable rectangles (since $\mathcal{E}$ is the smallest such σ-algebra). We first observe that if $R = A \times B$ is a measurable rectangle, then $R \in \mathcal{D}$ as R^x is either B or $\emptyset$ according to whether x is in A or not. As $\emptyset$ is a measurable rectangle, this also proves that $\emptyset \in \mathcal{D}$. To see that $\mathcal{D}$ is closed under complements, just recall that by f), $(E^c)^x = (E^x)^c$. Hence

$$E \in \mathcal{D} \iff E^x \in \mathcal{B} \iff (E^x)^c \in \mathcal{B} \iff (E^c)^x \in \mathcal{B} \iff E^c \in \mathcal{D}$$

The proof that $\mathcal{D}$ is closed under countable unions is similar: If $E_n \in \mathcal{D}$ for all n, then $(E_n)^x \in \mathcal{B}$ for all n, and hence $\bigcup_{n=1}^{\infty} (E_n)^x \in \mathcal{B}$. By f), $\bigcup_{n=1}^{\infty} (E_n)^x = \left(\bigcup_{n=1}^{\infty} E_n\right)^x$, and thus $\left(\bigcup_{n=1}^{\infty} E_n\right)^x \in \mathcal{B}$, which means that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}$.

i) We must show that $\mathcal{M}$ is closed under increasing and decreasing countable unions. Assume first that $\{E_n\}$ is an increasing sequence of sets in $\mathcal{M}$ and let $E = \bigcup_{n=1}^{\infty} E_n$. We must show that $E \in \mathcal{M}$. Since $E_n \in \mathcal{M}$ for all n, the functions $x \mapsto Q((E_n)^x)$ are $\mathcal{A}$-measurable for all n, and by f) and continuity of measure, $Q(E^x) = Q(\bigcup_{n=1}^{\infty} (E_n)^x) = \lim_{n\to\infty} Q((E_n)^x)$. Thus $x \mapsto Q(E^x)$ is the pointwise limit of the $\mathcal{A}$-measurable functions $x \mapsto Q((E_n)^x)$ and must be $\mathcal{A}$-measurable. Note also that since the sequence $x \mapsto Q((E_n)^x)$ increases to $x \mapsto Q(E^x)$, the Monotone Convergence Theorem tells us that

$$\int Q(E^x)\, dP(x) = \lim_{n\to\infty} \int Q((E_n)^x)\, dP(x)$$

Since $E_n \in \mathcal{M}$, we have $\int Q((E_n)^x)\, dP(x) = (P \times Q)(E_n)$, so

$$\int Q(E^x)\, dP(x) = \lim_{n\to\infty} (P \times Q)(E_n) = (P \times Q)(E)$$

by continuity of measure.

The proof for decreasing sequences is almost the same, but first we need to check that $\bigcap_{n\in\mathbb{N}} (E_n)^x = \left(\bigcap_{n\in\mathbb{N}} E_n\right)^x$. This is done just as for unions:

$$y \in \left(\bigcap_{n=1}^{\infty} E_n \right)^x \iff (x, y) \in \bigcap_{n=1}^{\infty} E_n \iff (x, y) \in E_n \text{ for all } n$$

$$\iff y \in (E_n)^x \text{ for all } n \iff y \in \bigcap_{n=1}^{\infty} (E_n)^x$$

Assume now that $\{E_n\}$ is an decreasing sequence of sets in $\mathcal{M}$ and let $E = \bigcap_{n=1}^{\infty} E_n$. We must show that $E \in \mathcal{M}$. Since $E_n \in \mathcal{M}$ for all n, the functions $x \mapsto Q((E_n)^x)$ are $\mathcal{A}$-measurable for all n, and by what we just proved and continuity of measure, $Q(E^x) = Q(\bigcap_{n=1}^{\infty} (E_n)^x) = \lim_{n\to\infty} Q((E_n)^x)$. Thus $x \mapsto Q(E^x)$ is the pointwise limit of the $\mathcal{A}$-measurable functions $x \mapsto Q((E_n)^x)$ and must be $\mathcal{A}$-measurable. Note also that since the sequence $x \mapsto Q((E_n)^x)$ converges to $x \mapsto Q(E^x)$, the Dominated Convergence Theorem (with the constant function $g = 1$ as the dominating function) tells us that

$$\int Q(E^x)\, dP(x) = \lim_{n\to\infty} \int Q((E_n)^x)\, dP(x)$$

Since $E_n \in \mathcal{M}$, we have $\int Q((E_n)^x)\, dP(x) = (P \times Q)(E_n)$, so

$$\int Q(E^x)\, dP(x) = \lim_{n\to\infty} (P \times Q)(E_n) = (P \times Q)(E)$$

by continuity of measure.

j) We are going to use the Monotone Class Theorem. If we can show that $\mathcal{R} \subseteq \mathcal{M}$, then $\mathcal{M}$ clearly contains the monotone class generated by $\mathcal{R}$. Since $\mathcal{R}$ is an algebra, the monotone class generated by $\mathcal{R}$ equals the σ-algebra $\mathcal{E}$ generated by $\mathcal{R}$, and hence $\mathcal{E} \subseteq M$. This means that

$$(P \times Q)(E) = \int Q(E^x)\, dP(x)$$

for all $E \in \mathcal{E}$ as we were supposed to show.

It only remains to show that $\mathcal{R} \subseteq \mathcal{M}$: If $R \in \mathcal{R}$, we have $R = \bigcup_{i=1}^{n} (R_i \times S_i)$ for disjoint, measurable rectangles $R_i \times S_i$. Note that

$$\mathbf{1}_R = \sum_{i=1}^{n} \mathbf{1}_{R_i \times S_i}$$

and hence

$$Q(R^x) = \int \mathbf{1}_{R^x}\, dQ(x) = \sum_{i=1}^{n} \int \mathbf{1}_{(R_i \times S_i)^x}\, dQ(x) = \sum_{i=1}^{n} \mathbf{1}_{R_i}(x) Q(S_i)$$

which shows that $x \mapsto Q(R^x)$ is $\mathcal{A}$-meassurable as it is a linear combination of $\mathcal{A}$-measurable simple functions. Moreover,

$$\int Q(R^x)\, dP(x) = \int \sum_{i=1}^{n} \mathbf{1}_{R_i}(x) Q(S_i)\, dP(x) = \sum_{i=1}^{n} Q(S_i) \int \sum_{i=1}^{n} \mathbf{1}_{R_i}(x)\, dP(x)$$

$$= \sum_{i=1}^{n} Q(S_i) P(R_i) = \sum_{i=1}^{n} (P \times Q)(R_i \times S_i) = (P \times Q)(R).$$

This shows that $R \in \mathcal{M}$.

k) First observe that we can write the result in i) as

$$E_{P\times Q}(\mathbf{1}_E) = E_P\left(\int \mathbf{1}_E(x,y)\,dQ(y)\right)$$

which shows that the result holds for indicator functions.

Let $\underline{U}_n$ be the n-th lower approximation to U. We have

$$E_{P\times Q}(\underline{U}_n) = \sum_{k=0}^{\infty} \frac{k}{2^n}(P\times Q)\left\{\frac{k}{2^n} < U \le \frac{k+1}{2^n}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{k}{2^n} E_{P\times Q}\left(\mathbf{1}_{\{\frac{k}{2^n}<U\le\frac{k+1}{2^n}\}}\right) = \sum_{k=0}^{\infty} \frac{k}{2^n} E_P\left(\int \mathbf{1}_{\{\frac{k}{2^n}<U\le\frac{k+1}{2^n}\}}(x,y)\,dQ(y)\right)$$

$$= E_P\left(\int \sum_{k=0}^{\infty} \frac{k}{2^n}\mathbf{1}_{\{\frac{k}{2^n}<U\le\frac{k+1}{2^n}\}}(x,y)\,dQ(y)\right) = E_P\left(\int \underline{U}_n(x,y)\,dQ(y)\right)$$

If we let $n\to\infty$, we see that $E_{P\times Q}(\underline{U}_n)\to E_{P\times Q}(U)$ by definition. As the sequence $\{\underline{U}_n\}$ is increasing pointwise to U, the sequence $\{\int \underline{U}_n(x,y)\,dQ(y)\}$ increases to $\int U(x,y)\,dQ(y)$ by the Monotone Convergence Theorem. Applying the Monotone Convergence Theorem again, this time to the sequence $\{\int \underline{U}_n(x,y)\,dQ(y)\}$, we get that

$$\lim_{n\to\infty} E_P\left(\int \underline{U}_n(x,y)\,dQ(y)\right) = E_P\left(\int U(x,y)\,dQ(y)\right)$$

Combining the two limits, we get

$$E_{P\times Q}(U) = E_P\left(\int U(x,y)\,dQ(y)\right)$$

as we were asked to show. (This result is usually called *Tonelli's Theorem*. If we add some integrability conditions, we can remove the assumption that U is nonnegative, and then we get *Fubini's Theorem*.)