UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Examination in: STK-MAT3710/4710 - Probability Theory.
Day of examination: Friday, December 3, 2021.

Examination hours: 15:00 - 19:00.

This problem set consists of 2 pages and 4 problems.

Appendices: Formula sheet.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

If there is a problem you can not solve, you may still use the result in the sequel. All answers have to be substantiated. For all problems, at each step, please explain the mathematical facts, which let you move on. Lack of explanation may cause a partial credit or no credits. Please write down your solutions completely in English.

[Problem 1] (9 points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ be sequences of random variables such that $\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$. Show that $\sum_{n=1}^{\infty} X_n$ and $\sum_{n=1}^{\infty} Y_n$ converge or diverge together.

[Problem 2]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and the random variable X is given by: $P(X = -5) = \frac{1}{4}$, $P(X = 3) = \frac{1}{4}$, and $P(X = 1) = \frac{1}{2}$.

a) (9 points) Find the Taylor expansion of the characteristic function of X.

b) (9 points) Assume that $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of independent random variables with the same distribution as X. Find the characteristic function of:

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$

c) (9 points) Use the result in **(b)** to show directly that S_n converges in distribution to a normal distribution (you are not allow to use a version of the Central Limit Theorem).

[Problem 3]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a sub-sigma field of $\mathcal{F}$. Suppose that $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ be a martingale and $E[X_n^2] < \infty$ for all $n \in \mathbb{N}_0$. Moreover, assume that $n < r < m$ for all $m, n, r \in \mathbb{N}_0$.

a) (9 points) Show that for all $j > n$ ($j \in \mathbb{N}_0$), $E[X_j X_n] = E[X_n^2]$.

b) (5 points) Show that $E[(X_m - X_r)X_n] = 0$.

c) (9 points) Show that $E[(X_m - X_r)^2|\mathcal{F}_n] = E[X_m^2|F_n] - E[X_r^2|F_n]$.

d) (9 points) Suppose that there exists a constant K such that $E[X_n^2] \leq K$ for all $n \in \mathbb{N}_0$. Show that $\{X_n\}_{n\in\mathbb{N}_0}$ is mean-square Cauchy convergent.

[Problem 4]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a sub-sigma field of $\mathcal{F}$. Suppose that $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ be a martingale and let K be a constant. Define T and S with $\lambda_2 < 0 < \lambda_1$ as follows:

$$T(\omega) = \begin{cases} \inf\{n \in \mathbb{N}_0 : X_n \geq \lambda_1\}, & \text{if such n exists,} \\ N, & \text{if no such } n \text{ exists,} \end{cases}$$

and

$$S(\omega) = \begin{cases} \inf\{n \in \mathbb{N}_0 : X_n \leq \lambda_2\}, & \text{if such } n \text{ exists,} \\ N, & \text{if no such } n \text{ exists.} \end{cases}$$

a) (9 points) Prove that $\gamma = \max(T, S)$ is a *bounded* stopping time.

b) (10 points) Let us define $X_n^+ = \max(X_n, K)$ and assume that $K < \lambda_2$. Let $E[X_S^+] = L$ be (L is a constant), then, is $E[X_\gamma^+]$ less than L, equal to L or greater than L? Justify your reasoning.
Hint: $X^+ = \max(X, K)$ is a nondecreasing convex function of X.

c) (13 points) Let assume $K = 0$. Suppose that $M_n = \lim_{m\to\infty} E[X_{m+n}^+|\mathcal{F}_n]$, ($m \geq 0$), exists almost surely, moreover, M_n is integrable and $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}_0$. Prove that M_n is a martingale with respect to $\mathcal{F}_n$.

GOOD LUCK!