UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Soltions of the Examination, STK-MAT3710/4710 - Probability Theory.

[Problem 1] (9 points) **Solution:** By the first Borel-Cantelli lemma, $P(X_n \neq Y_n, i.o.) = 0$. Hence, $X_n = Y_n$ for all, just not for finitely many values of n, almost surely. Off an event of probability zero, the sequences are identical for all large n. Hence, both converge or diverge together.

[Problem 2]

a) (9 points) **Solution:** $E[X] = -5 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} = 0$.
$E[X^2] = 25 \cdot \frac{1}{4} + 9 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} = 9$.
Then, by Taylor expansion formula of the characteristic function:
$\phi_X(t) = 1 - \frac{9}{2}t^2 + o(t^2)$.
If you have written :

$$\phi_X(t) = \sum_{k=0}^{n} \frac{1}{k!} E[X^k](it)^k + o(t^n) = \sum_{k=0}^{n} \frac{(-5)^k + 3^k + 2}{4k!}(it)^k + o(t^n)$$

is also accepted.

b) (9 points) **Solution:**

$$\begin{aligned}
\phi_{S_n}(t) &= E[e^{itS_n}] = E[e^{it(\frac{X_1+X_2+\ldots+X_n}{\sqrt{n}})}] \\
&= E[e^{\frac{it}{\sqrt{n}}X_1} \cdot e^{\frac{it}{\sqrt{n}}X_2} \cdots e^{\frac{it}{\sqrt{n}}X_n}] \\
&= E[e^{\frac{it}{\sqrt{n}}X_1}] \cdot E[e^{\frac{it}{\sqrt{n}}X_2}] \cdots E[e^{\frac{it}{\sqrt{n}}X_n}], \quad (X_n\text{'s are independent}) \\
&= (E[e^{\frac{it}{\sqrt{n}}X_1}])^n \quad (X_n\text{'s are identical distributed}) \\
&= (\phi_{X_1}(\frac{t}{\sqrt{n}}))^n = (\phi_X(\frac{t}{\sqrt{n}}))^n \\
&= (1 - \frac{9}{2}\frac{t^2}{n} + o(\frac{t^2}{n}))^n.
\end{aligned}$$

Last step runs since X_n's and X have the same distribution (so they have same characteristic function).

c) (9 points) **Solution:** Let us define $z_n = \frac{-9}{2}t^2 + o(t^2)$, hence by the Lemma 6.34,

$$\lim_{n\to\infty}(1 + \frac{z_n}{n})^n = e^{\frac{-9}{2}t^2}, \quad (z_n \to \frac{-9}{2}t^2, \ (n \to \infty)).$$

Since $e^{\frac{-9}{2}t^2}$ is continuous at $t = 0$ and $\phi_{S_n}(t) \to e^{\frac{-9}{2}t^2}$ $(n \to \infty)$, which is the characteristic function of a random variable with Normal distribution, by Levy continuity theorem, S_n converges in distribution to a normal distribution with $N(0, 9)$.

[Problem 3]
a) (9 points) **Solution:**

$$\begin{aligned} E[X_j X_n] &= E[E[X_j X_n|\mathcal{F}_n]] \quad (n < j,\ X_n \text{ is } \mathcal{F}_n\text{-measurable}) \\ &= E[X_n E[X_j|\mathcal{F}_n]] = E[X_n^2] \quad (X_n \text{ is a martingale wrt } \mathcal{F}_n). \end{aligned}$$

First step runs via the rule of conditional expectation of a random variable is equal to its expectation.

b) (5 points) **Solution:**

$$E[(X_m - X_r)X_n] = E[X_m X_n] - E[X_r X_n] = E[X_n^2] - E[X_n^2] = 0 \quad \text{by (a) and expectation is linear}$$

c) (9 points) **Solution:**

$$\begin{aligned} E[(X_m - X_r)^2|\mathcal{F}_n] &= E[X_m^2|\mathcal{F}_n] - 2E[X_m X_r|\mathcal{F}_n] + E[X_r^2|\mathcal{F}_n] \quad \text{conditional expectation is linear} \\ &= E[X_m^2|\mathcal{F}_n] - E[X_r^2|\mathcal{F}_n], \quad \text{(by 1)}. \end{aligned}$$

and

$$\begin{aligned} E[X_m X_r|\mathcal{F}_n] &= E[E[X_m X_r|\mathcal{F}_r]|\mathcal{F}_n] \quad \text{by tower property, } n < r, \text{ and } r < m \\ &= E[X_r E[X_m|\mathcal{F}_r]|\mathcal{F}_n] \quad (X_r \text{ is } \mathcal{F}_r\text{-measurable}) \\ &= E[X_r X_r|\mathcal{F}_n] = E[X_r^2|\mathcal{F}_n] \quad (X_n \text{ is a martingale wrt } \mathcal{F}_n) \quad (1) \end{aligned}$$

d) (9 points) **Solution:**
By taking expectation of both sides of the equation obtained in part (c) and by the linearity of conditional expectation, we get:

$$\begin{aligned} &0 \le E[E[(X_m - X_r)^2|\mathcal{F}_n]] = E[E[X_m^2|\mathcal{F}_n] - E[X_r^2|\mathcal{F}_n]] = E[E[(X_m^2 - X_r^2)|\mathcal{F}_n]] \\ &0 \le E[(X_m - X_r)^2] = E[X_m^2 - X_r^2], \quad r < m. \end{aligned}$$

Observe that since $E[X_m^2 - X_r^2] \ge 0$ for $r < m$, $E[X_m^2] \ge E[X_r^2]$, for all $r < m$. Hence, $\left\{E(X_n^2) : n \ge 1\right\}$ is nondecreasing and by hypothesis it is bounded. Therefore, $\left\{E(X_n^2) : n \ge 1\right\}$ is convergent. Then,

$$\lim_{m\to\infty} E[X_m^2] = \lim_{r\to\infty} E[X_r^2], \quad \text{(If limit exists, it is unique.)}$$

By Squeeze Theorem, $\{X_n\}_{n\in\mathbb{N}_0}$ is Cauchy convergent.

[Problem 4]

a) (9 points) **Solution:**

Both S and T are bounded First Hitting Times, hence they are stopping times. The natural filtration generated by X_n can be considered as a sub-filtration of $\mathcal{F}_n$, hence both S and T are $\mathcal{F}_n$-measurable and both $\{T \leq n\} \in \mathcal{F}_n$ and $\{S \leq n\} \in \mathcal{F}_n$. Then,

$$\{\gamma = \max(T, S) \leq n\} = \{T \leq n\} \cap \{S \leq n\} \in \mathcal{F}_n,$$

σ-algebras are closed under finite intersection. Since, both S and T are bounded, maximum of their upper bound is an upper bound for γ.

b) (10 points) **Solution:**

Let us call $\max(X_n, K) = X_n^+ = \Phi(X_n)$, since Φ is a convex function and X_n is an $\mathcal{F}_n$-martingale; hence by Jensen's inequality:

$$E[X_{n+1}^+|\mathcal{F}_n] = E[\Phi(X_{n+1})|\mathcal{F}_n] \geq \Phi(E[X_{n+1}|\mathcal{F}_n]) = \Phi(X_n) = X_n^+.$$

Hence, X_n^+ is an $\mathcal{F}_n$-submartingale.
Observe that since $K < \lambda_2$, replacing X_n^+ with X_n in the definitions of S or T does not matter. By hypothesis, $\gamma \geq S$ and both are bounded stopping-times, hence, by Theorem 9.9, since X_n^+ is an $\mathcal{F}_n$-submartingale, (X_γ^+, X_S^+) is an $(\mathcal{F}_S, \mathcal{F}_\gamma)$ submartingale. Then, by applying expectation to both sides of the inequality:

$$\begin{aligned}
&E[X_\gamma^+|\mathcal{F}_S] \geq X_S^+ \\
&E[E[X_\gamma^+|\mathcal{F}_S]] \geq E[X_S^+] \\
&E[X_\gamma^+] \geq E[X_S^+] = L.
\end{aligned}$$

c) (13 points) **Solution:**

Since $K = 0$, $U_m = E[X_{m+n}^+|\mathcal{F}_n]$, $m \geq 0$ is a nonnegative sequence of random variables. (Remember that conditional expectation of a random variable is a random variable as well.) Moreover, by Tower property:

$$U_{m+1} = E[X_{m+n+1}^+|\mathcal{F}_n] = E[E[X_{m+n+1}^+|\mathcal{F}_{n+m}]|\mathcal{F}_n] \geq E[X_{n+m}^+|\mathcal{F}_n] = U_m,$$

since X_n^+ is an $\mathcal{F}_n$-submartingale. We see that for all $m \geq 0$, $U_{m+1} \geq U_m$, then U_m is nondecreasing (or increasing). So we are allowed to use Monotone Convergence Theorem for conditional expectation. Then,

$$\begin{aligned}
E[M_{n+1}|\mathcal{F}_n] &= E[\lim_{m\to\infty} E[X_{m+n+1}^+|\mathcal{F}_{n+1}]|\mathcal{F}_n] \\
&= \lim_{m\to\infty} E[E[X_{m+n+1}^+|\mathcal{F}_{n+1}]|\mathcal{F}_n] \\
&= \lim_{m\to\infty} E[X_{m+n+1}^+|\mathcal{F}_n] = M_n, \quad \text{by Tower property.}
\end{aligned}$$

Note that last step runs also by the fact that if limit exists, it has to be unique and all subsequences approach to same point.