STK-MAT3710/4710 Probability Theory

Mandatory assignment 1 of 1

November 15, 2021

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1 Solutions

[Problem 1] i) $\emptyset \in \mathcal{A}$. Indeed, $\mathbb{1}_\emptyset = 0 \in \mathcal{G}$. ($f(x) = 0$, constant function.) 4
ii) For any $A \in \mathcal{A}$, $A^c \in \mathcal{A}$. Indeed, $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A \in \mathcal{G}$. ($\mathcal{G}$ is closed under $f - g$.) 4
iii) For any $A, B \in \mathcal{A}$, $\mathbb{1}_{A\cap B} = \mathbb{1}_A \cdot \mathbb{1}_B \in \mathcal{G}$. Hence, $\mathcal{A}$ is closed under finite intersection. 4
$\mathcal{A}$ is an algebra. Then, let us show that it is also a monotone class.
iv) For any increasing sequence of sets $A_1, A_1, \dots \in \mathcal{A}$, then, $\lim_{n\to\infty} A_n = \bigcup_{n=1}^\infty A_n = A$ exists. Hence,

$$\mathbb{1}_{\bigcup_{n=1}^\infty A_n} = \lim_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A \in \mathcal{G}, \quad 3$$

since $\mathcal{G}$ is closed under pointwise convergence.
v) For any decreasing set sequence $A_1, A_1, \dots \in \mathcal{A}$, then, $\lim_{n\to\infty} A_n = \bigcap_{n=1}^\infty A_n = A$ exists. Hence,

$$\mathbb{1}_{\bigcap_{n=1}^\infty A_n} = \lim_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A \in \mathcal{G}, \quad 3$$

since $\mathcal{G}$ is closed under pointwise convergence.
Since $\mathcal{A}$ is an algebra and monotone class, then it is a sigma algebra. 2

20 **[Problem 2]** Suppose that $d_1(F, G) = 0$, then

$$F(x) \le \lim_{\epsilon\downarrow 0}(G(x+\epsilon)+\epsilon) = G(x) \quad (G \text{ is a distribution function, hence it is right continuous}),$$

4 3

and

$$F(y) \ge \lim_{\epsilon\downarrow 0}(G(y-\epsilon)-\epsilon) = G(y-) \quad (G \text{ is a distribution function, then it has left limits}).$$

4 3

2 $G(y-) \ge G(x)$ if y>x (G is a distribution function, so increasing) and F is a distribution function so right-continuous, then taking the limit $y \downarrow x$, we get

2

$$F(x) \ge \lim_{y\downarrow x} G(-y) \ge G(x),$$

2

implies that $F(x) = G(x)$ for all x.

15 **[Problem 3]** Suppose $d_2(X_n, X) \overset{pointwise}{\longrightarrow} 0$. Then, fix $a \in \mathbb{R}$, and let u (3) be the indicator function of the interval $(-\infty, a]$ (2). Then,

$$0 \le |F_n(a) - F_X(a)| = |P(X_n \le a) - P(X \le a)| \le \sup_{u:\|u\|_\infty = 1} |E(u(X_n)) - E(u(X))| \overset{pointwise}{\longrightarrow} 0.$$

2 5 3

20/ **[Problem 4]** Let us assume

$$\sum_{n\in\mathbb{N}} P(|X_n - X| > \epsilon) < \infty, \qquad 3$$

2

then, by the first Borel-Cantelli Lemma, $|X_n - X| > \epsilon$ occurs only finitely often with probability 1, for all $\epsilon > 0$. This implies that $X_n \overset{a.s.}{\to} X$. 3

Suppose conversely that $X_n \overset{a.s.}{\to} X$. Then, $X_n \xrightarrow{prob} X$. 3 By hint, X is almost surely constant. Hence, $X_n \overset{a.s.}{\to} c$, 1 $c \in \mathbb{R}$. Then, by formal (limit) definition of almost convergence, for any $\epsilon > 0$, only finitely many of the independent events $|X_n - c| > \epsilon$ occur with probability 1. Using second Borel-Cantelli lemma, so 3

3 2

$$\sum_{n\in\mathbb{N}} P(|X_n - c| > \epsilon) < \infty.$$

25/ **[Problem 5] a)** Since $X = X^+ - X^-$, it is enough to show for $X \geq 0$. By Equation (1): 13

1 1

$$\int_A X dP = \int_\Omega \mathbb{1}_A X dP = \int_\Omega \lim_{n\to\infty} \mathbb{1}_A X_n dP$$

+1

$$4 \text{ (by 2)} = \lim_{n\to\infty} \int_\Omega \mathbb{1}_A \left\{ \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{1}_{\{\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\}} + n\mathbb{1}_{\{X\geq n\}} \right\} dP$$

$$\text{(by 1)} = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left(\left\{ \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n} \right\} \cap A \right) + nP\Big(\{X \geq n\} \cap A \Big) \right\}$$

2 2

$$2 \text{ (by 3-4)} = 0$$

(1) $\mathbb{1}_{C\cap D} = \mathbb{1}_C \cdot \mathbb{1}_D$
(2) $\mathbb{1}_A X_n \uparrow \mathbb{1}_A X$ and $\mathbb{1}_A X_n \geq 0$, hence Monotone Convergence Theorem is applicable.
(3) A is P-negligible; hence $P(A) = 0$.
(4) $B \subseteq C$, then $P(B) \leq P(C)$.

b) 9/

1 1

$$\int_A X dP = \int_\Omega \mathbb{1}_A X dP = \int_\Omega \mathbb{1}_{\bigcup_{n=\mathbb{N}} A_n} X dP \quad 1$$

$$3 \text{ (by 1,3)} = \int_\Omega \lim_{n\to\infty} \mathbb{1}_{A_n} X^+ dP - \int_\Omega \lim_{n\to\infty} \mathbb{1}_{A_n} X^- dP$$

$$3 \text{ (by 2)} = \lim_{n\to\infty} \int_\Omega \mathbb{1}_{A_n} X^+ dP - \lim_{n\to\infty} \int_\Omega \mathbb{1}_{A_n} X^- dP$$

$$= \lim_{n\to\infty} \int_\Omega \mathbb{1}_{A_n} (X^+ - X^-) \quad 1$$

$$= \lim_{n\to\infty} \int_\Omega \mathbb{1}_{A_n} X dP \quad 1$$

$$= \lim_{n\to\infty} \int_{A_n} X dP \quad 1$$

(1) $X = X^+ - X^-$ (1)
(2) $\mathbb{1}_{A_n} X^+ \uparrow \mathbb{1}_A X^+$ and $\mathbb{1}_{A_n} X^- \uparrow \mathbb{1}_A X^-$ and both of the sequences are with nonegative terms, hence Monotone convergence theorem is applicable. (3)
(3) Note that $\int_\Omega \mathbb{1}_{A_n} X dP = E[\mathbb{1}_{A_n} X]$, hence it is allowed to separate limit. (2)