STK-MAT3710/4710 Probability Theory
Mandatory assignment 1 of 1

October 21, 2021

1 Submission deadline

Thursday 28th October 2021, 14:30 in Canvas (canvas.uio.no).

2 Instructions

If you are taking STK-MAT3710, you can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LATEX). If you are taking STK-MAT4710, you have to write the assignment in LATEX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. You need a 60 % score to pass this assignment. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

3 Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:studieinfo@math.uio.no) well before the deadline.

4 Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

5 Problems

Problem 1. Suppose $\mathcal{G}$ is a collection of real valued functions on Ω such that $\mathcal{G}$ contains constant functions and for all $f,\ g \in \mathcal{G}$ and $c \in \mathbb{R}$, $f+g,\ fg,\ cf$ are in $\mathcal{G}$. Moreover, suppose that for all $f_n \in \mathcal{G},\ n \in \mathbb{N}$, whenever $f_n \to f$ pointwise, then $f \in \mathcal{G}$.

Let us define the indicator function $\mathbb{1}_A$ as follows:

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \text{if } \ \omega \in A \\ 0, & \text{if } \ \omega \notin A \end{cases}$$

Prove that $\mathcal{A} = \{A \subseteq \Omega : \mathbb{1}_A \in \mathcal{G}\}$ is a σ-algebra.

(***Hint:*** $\mathbb{1}_{A\cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$)

Problem 2. Let F and G be distribution functions and let define $d_1(F,G)$ as follows:

$$d_1(F,G) = \inf\{\epsilon > 0 : G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\}.$$

Show that if $d_1(F,G) = 0$, then $F(x) = G(x)$, for all $x \in \mathbb{R}$.

Problem 3. Let X and Y be two random variables and let define:

$$d_2(X,Y) = \sup_{u:\|u\|_\infty = 1} |E(u(X)) - E(u(Y))|,$$

where the supremum is over all measurable functions $u : \mathbb{R} \to \mathbb{R}$ such that $\|u\|_\infty = \sup_x |u(x)|$, which satisfies $\|u\|_\infty = 1$.

Let $X_1, X_2, \ldots$ be a sequence of random variables. Show that if $d_2(X_n, X) \to 0,\ (n \to \infty)$, then $X_n \to X,\ (n \to \infty)$ in distribution.

Problem 4. Let $X_1, X_2, \ldots$ be a sequence of independent random variables and let X be a random variable.

For all $\epsilon > 0$, $\quad \sum_{n\in\mathbb{N}} P(|X_n - X| > \epsilon) < \infty$ if and only if $X_n \to X\ (n \to \infty)$ almost surely.

(***Hint:*** Let $X_n,\ n \in \mathbb{N}$ be a sequence of random variables, which converges in probability to the random variable X. Then, X is almost surely constant.)

Problem 5. For the extended real line, $\overline{\mathbb{R}} = [-\infty, \infty]$, the Borel σ-field can be defined as follows:

$$\overline{\mathcal{B}} = \mathcal{B}(\overline{\mathbb{R}}) = \sigma(\{[-\infty, x],\ x \in \overline{\mathbb{R}}\}).$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space. An *extended real random variable* (errv) X is an $\overline{\mathcal{B}}$-measurable mapping of Ω into $\overline{\mathbb{R}}$.

For an errv (general) X, $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$ are called the positive and negative parts of X. Note that both X^+ and X^- are positive and $X = X^+ - X^-$.

We define

$$E[X] = \int_\Omega X dP = E[X^+] - E[X^-],$$

whenever $E[X^+] < \infty$ **or** $E[X^-] < \infty$. Then, the errv X is called *quasi-integrable* and $E[X] \in \overline{\mathbb{R}}$.

Here, we assume that $\infty -$ (finite value) $= \infty$, (finite value) $- \infty = -\infty$, and $0 \cdot \infty = 0$ (not indeterminate). In all these cases, we assume that expectation exists with a zero or infinite value. But, note that, $\infty - \infty$ is indeterminate, and in this case, we simply say that expectation does not exist.

Moreover, $X \to \overline{\mathbb{R}}$ is an errv if and only if there exists a sequence of simple random variables $\{X_n,\ n \in \mathbb{N}\}$ such that X_n converges to X pointwise.

Hence, let X be a *non-negative* errv, then there exists an *increasing* sequence of simple random variables X_n such that

$$X_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{1}_{\left\{\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}\right\}} + n\mathbb{1}_{\{X \geq n\}}, \tag{1}$$

which converges to X pointwise, i.e., $X_n \uparrow X$ pointwise.

Show that for a quasi-integrable errv X, which is defined on $(\Omega, \mathcal{A}, P)$, the following properties hold:

a) By using (1), prove that if $A \in \mathcal{A}$ is a P-negligible set, then $\int_A X dP = 0$.

(***Hint:*** A set $A \in \Omega$ is called P-negligible if it is contained in an event $N \in \mathcal{A}$; i.e., $A \subset N$, and $P(N) = 0$.)

b) If $\{A_n, n \geq 1\}$ is an increasing sequence in $\mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} A_n = A$, then

$$\lim_{n\to\infty} \int_{A_n} X dP = \int_A X dP.$$

!!! For all problems, at each step, please explain the mathematical facts which let you move on. Lack of explanation may cause a partial credit or no credits.

GOOD LUCK.