Abel Summation. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers, and let $s_n = \sum_{k=1}^{n} a_k$. Then

$$\sum_{n=1}^{N} a_n b_n = s_N b_N + \sum_{n=1}^{N-1} s_n (b_n - b_{n+1}).$$

If $\lim_{N\to\infty} s_N b_N = 0$, then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} s_n (b_n - b_{n+1})$$

in the sense that either the two series both diverge or they converge to the same limit.

Proof: Note that $a_n = s_n - s_{n-1}$ for $n > 1$, and that this formula even holds for $n = 1$ if we define $s_0 = 0$. Hence

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} (s_n - s_{n-1}) b_n = \sum_{n=1}^{N} s_n b_n - \sum_{n=1}^{N} s_{n-1} b_n$$

Changing the index of summation and using that $s_0 = 0$, we see that $\sum_{n=1}^{N} s_{n-1} b_n = \sum_{n=1}^{N-1} s_n b_{n+1}$. Putting this into the formula above, we get

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} s_n b_n - \sum_{n=1}^{N-1} s_n b_{n+1} = s_N b_N + \sum_{n=1}^{N-1} s_n (b_n - b_{n+1})$$

and the first part of the lemma is proved. The second follows by letting $N \to \infty$.