Section 2.5: Expectations I

Problem: What is $E[X]$ = average value of X

Recall: A series $\sum_{n=1}^{\infty} a_n$ is convergent if the partial sums $\sum_{n=1}^{N} a_n$ converge to a number S when $N \to \infty$. If so, we write $S = \sum_{n=1}^{\infty} a_n$

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges (this implies that $\sum_{n=1}^{\infty} a_n$ converges)

If $\sum_{n=1}^{\infty} a_n$ converges but not absolutely, then it is conditionally convergent

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conditionally convergent.

Theorem: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, the sum is independent of the order of the a_n's.

Definition: A random variable $X : \Omega \to \mathbb{R}$ is discrete if there is a countable set $Q = \{x_1, x_2, x_3, \ldots\}$ such that

$$P[X(\omega) \in Q] = 1$$

Definition: Assume that X is a discrete random variable taking the distinct values $x_1, x_2, x_3, \ldots$

We say that X is integrable if

$$\sum_i |x_i| P[X = x_i] < \infty$$

If X is integrable, we define the expectation of X by

$$E[X] = \sum_i x_i P[X = x_i]$$

Lemma: Assume that X is a discrete integrable r.v. and that we have sets $\Delta_1, \Delta_2, \Delta_3, \ldots$ such that the Δ_i's are disjoint, $P(\bigcup_i \Delta_i) = 1$ and X is constant on each Δ_i. Then

$$E[X] = \sum_i \lambda_i P(\Delta_i)$$

where λ_i is X's value on Δ_i

Proof: $E[X] = \sum_i x_i P[X = x_i]$

$= \sum x_i P[\Delta_{i_1} \cup \Delta_{i_2} \cup \ldots]$

$= \sum_i x_i [P(\Delta_{i_1}) + P(\Delta_{i_2}) + \cdots + P(\Delta_{i_n}) \cdots]$

$= \sum_i [\lambda_{i_1} P(\Delta_{i_1}) + \lambda_{i_2} P(\Delta_{i_2}) + \cdots + \lambda_{i_n} P(\Delta_{i_n})]$

$= \sum_i \lambda_i P(\Delta_i)$

Theorem: If X, Y are two discrete, integrable r.v., then $aX + bY$ is an integrable r.v. for all $a, b \in \mathbb{R}$ and

$$E[aX + bY] = aE[X] + bE[Y]$$

Proof: Let $x_1, x_2, \ldots$ be the values of X and $y_1, y_2, \ldots$ be the values of Y. Let

$$\Delta_{ij} = \{\omega : X(\omega) = x_i \text{ and } Y(\omega) = y_j\}$$

Note that

$$\sum_{i,j} |a x_i + b y_j| P(\Delta_{ij}) = E[|aX + bY|]$$

$$\le$$

$$\sum_{i,j} (|a||x_i| + |b||y_j|) P(\Delta_{ij})$$

$= |a| \sum_{i,j} |x_i| P(\Delta_{ij}) + |b| \sum_{i,j} |y_j| P(\Delta_{ij})$

$= |a| E[|X|] + b E[|Y|] < \infty.$

Similarly,

$$E[aX + bY] = \sum_{i,j} (a x_i + b y_j) P(\Delta_{ij})$$

$= a \sum_{i,j} x_i P(\Delta_{ij}) + b \sum_{i,j} y_j P(\Delta_{ij})$

$= a E[X] + b E[Y].$

Prop: If X and Y are two discrete r.v. with $|X| \le |Y|$ and Y is integrable, then X is integrable.

Proof: $\Delta_{ij} = \{\omega : X(\omega) = x_i \text{ and } Y(\omega) = y_j\}$

We have

$$\sum |x_i| P(\Delta_{ij}) \le \sum |y_j| P(\Delta_{ij}) < \infty$$

since Y is integrable.

Prop: If X, Y are integrable, discrete r.v. and $X \le Y$, then

$$E[X] \le E[Y].$$

Proof: $0 \le E[Y - X] = E[Y] - E[X]$

Hence $E[X] \le E[Y]$.

Corollary: X discrete, integrable. Then

$$E[X] \le E[|X|]$$

Variance: $\mathrm{Var}(X) = E[(X - E[X])^2]$