Exercises September 7th

Alternative descriptions of random variables

a) $$(-\infty, x) = \bigcup_{n \in \mathbb{N}} (-\infty, x - \tfrac{1}{n}]$$

Assume that $y \in (-\infty, x)$. This means that $y < x$, and hence there is a $n \in \mathbb{N}$ such that $y < x - \frac{1}{n}$. Hence $y \in (-\infty, x-\frac{1}{n}]$, i.e. $y \in \bigcup_{n \in \mathbb{N}} (-\infty, x-\frac{1}{n}]$

Assume next that $y \in \bigcup_{n \in \mathbb{N}} (-\infty, x-\frac{1}{n}]$. Then must be an $n \in \mathbb{N}$ s.t. $y \in (-\infty, x-\frac{1}{n}]$. Hence $y \leq x - \frac{1}{n} < x$, which means that $y \in (-\infty, x)$.

Next $$(-\infty, x] = \bigcap_{n \in \mathbb{N}} (-\infty, x + \tfrac{1}{n})$$

Assume $y \in (-\infty, x]$. Then for all n, $y \leq x < x + \frac{1}{n}$, hence $y \in (-\infty, x+\frac{1}{n})$.

Thus $y \in \bigcap_{n \in \mathbb{N}} (-\infty, x+\frac{1}{n})$

Assume next that $y \in \bigcap_{n \in \mathbb{N}} (-\infty, x+\frac{1}{n})$

This means that $y < x + \frac{1}{n}$ for all n, and hence $y \leq x$, hence $y \in (-\infty, x]$.

b) Prove that TFAE:

(i) X is a r.v. $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$ for all x

(ii) $\{\omega : X(\omega) < x\} \in \mathcal{F}$ for all x

(iii) $\{\omega : X(\omega) \geq x\} \in \mathcal{F}$ for all x

(iv) $\{\omega : X(\omega) > x\} \in \mathcal{F}$ for all x

Proof: (i) $\Leftrightarrow$ (ii) Assume (i). Then

$$\{\omega : X(\omega) < x\} = \bigcup_{n \in \mathbb{N}} \{\omega : X(\omega) \leq x - \tfrac{1}{n}\} \in \mathcal{F}.$$

(ii) $\Rightarrow$ (i) Assume (ii). Then

$$\{\omega : X(\omega) \leq x\} = \bigcap_{n \in \mathbb{N}} \{\omega : X(\omega) < x + \tfrac{1}{n}\} \in \mathcal{F}.$$

(ii) $\Rightarrow$ (iii)

$$\{\omega : X(\omega) \geq x\} = \underbrace{\{\omega : X(\omega) < x\}}^{c} \in \mathcal{F}$$

(iii) $\Rightarrow$ (ii)

$$\{\omega : X(\omega) < x\} = \underbrace{\{\omega : X(\omega) \geq x\}}_{\mathcal{F}}^{c} \in \mathcal{F}.$$