Last time:

Prop: Assume that X, Y are random variables. Then the following are also random variables.

V (i) aX for all $a \in \mathbb{R}$.

V (ii) $X+Y$

(iii) XY

V (iv) $Z(\omega) = \begin{cases} \dfrac{X(\omega)}{Y(\omega)} & \text{whenever } Y(\omega) = 0 \\ 0 & \text{else.} \end{cases}$

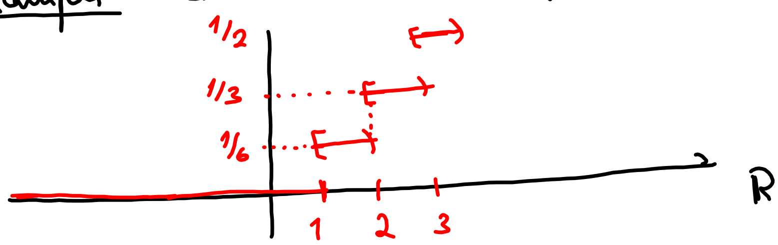
Proof (iii) First prove that if Z is a r.v., then Z^2 is.

Note
$$\{\omega : Z^2(\omega) \leq \alpha\} = \underbrace{\{\omega : Z(\omega) \leq \sqrt{\alpha}\}}_{\in \mathcal{A}} \cap \underbrace{\{\omega : Z(\omega) \geq -\sqrt{\alpha}\}}_{\in \mathcal{A}} \in \mathcal{A}$$

Note
$$XY = \frac{1}{2}\Big[\underbrace{(X+Y)^2}_{r.v} - \underbrace{X^2}_{r.v} - \underbrace{Y^2}_{r.v}\Big] \quad \text{is a r.v.}$$

Definition: Assume that X is a random variable. The distribution function F of X is the function $F : \mathbb{R} \to [0,1]$ given by
$$F(x) = P\underbrace{\{\omega : X(\omega) \leq x\}}_{\in \mathcal{A}}$$

Example: X is the result of throwing a die: 1, 2, 3, 4, 5, 6



Properties of distribution functions

(i) $0 \le F(x) \le 1$

(ii) If $x < y$, then $F(x) \le F(y)$

(iii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

(iv) Right continuity: $F(x) = \lim_{y \downarrow x} F(y)$

(v) Left limits: $F(x-) = \lim_{y \uparrow x} F(y) = P[\omega : X(\omega) < x]$

(vi) F has at most a countable number of discontinuous

Proof: (i) is obvious.

(ii) $F(x) = P\{\omega : X(\omega) \le x\} \le P\{\omega : X(\omega) \le y\} = F(y)$

(iii) $\lim_{x \to -\infty} F(x) = \lim_{n \to \infty} F(-n) = \lim_{n \to \infty} P\{\underbrace{\omega : X(\omega) \le -n}_{\text{decreasing with intersection } \emptyset}\}$

$\underset{\text{cont. of measure}}{=} P(\emptyset) = 0$

$\lim_{x \to \infty} F(x) = \lim_{n \to \infty} F(n) = \lim_{n \to \infty} P\{\underbrace{\omega : X(\omega) \le n}_{\text{increasing with union } \Omega}\}$

$\underset{\text{cont. of measures}}{=} P(\Omega) = 1$

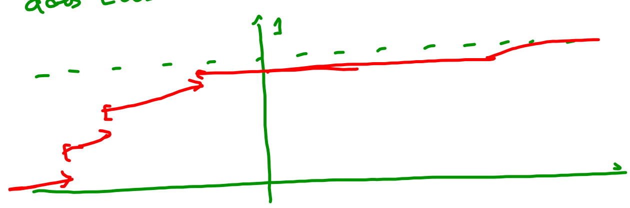
(iv) $\lim_{y \downarrow x} F(y) = \lim_{n \to \infty} F(x + \frac{1}{n}) = \lim_{n \to \infty} P\overbrace{\{\omega : X(\omega) \le x + \tfrac{1}{n}\}}^{\text{decreasing with intersection } (-\infty, x]}$

$\underset{\text{cont of measure}}{=} P[X(\omega) \in (-\infty, x]] = P[\omega : X(\omega) \le x] = F(x)$.

(v) $\lim_{y \uparrow x} F(y) = \lim_{n \to \infty} F(x - \frac{1}{n}) = \lim_{n \to \infty} P\overbrace{\{\omega : X(\omega) \le x - \tfrac{1}{n}\}}^{\text{increasing with union } \{\omega : X(\omega) < x\}}$

$\underset{\text{cont of measures}}{=} P[\omega : X(\omega) < x] \overset{\text{def}}{=} F(x-)$

What does this mean:



(vi) Why does F have at most a countable number of discontinuities?

Note that F can have at most n jumps of size larger $\frac{1}{n}$.

List the jumps by size

$x_1, x_2, x_3, \dots \quad , \quad x_{n_1}$

and you will get them all

Problems

Page 4, ex 1.2: Problem: How do we prove that $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$?

Observation: Assume that we can prove that for any $G \in \mathcal{G}$, we have $G \in \sigma(\mathcal{F})$; i.e. $\mathcal{G} \subseteq \sigma(\mathcal{F})$. Then $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$ (since $\sigma(\mathcal{G})$ is the smallest σ-algebra containing $\mathcal{G}$)

Similarly, if $\mathcal{F} \subseteq \sigma(\mathcal{G})$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G})$

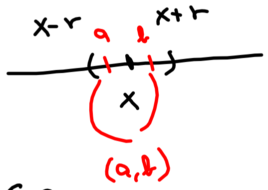
Proposition: Let $\mathcal{O}$ be the collection of all open sets and $\mathcal{I}$ the collection of all open intervals. Then

Borel $= \sigma(\mathcal{O}) = \sigma(\mathcal{I})$

Proof: Assume that $I \in \mathcal{I}$, then $I \in \mathcal{O} \subseteq \sigma(\mathcal{O})$. Hence $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{O})$

Assume that $O \in \mathcal{O}$, i.e. O is an open set. Need to prove $O \in \sigma(\mathcal{I})$. Observe that if $x \in O$, then there is an interval (a,b) with rational endpoints such that

$x \in (a,b) \subseteq O$



Claim that

$$O = \bigcup \{(a,b) : \text{when } a, b \text{ are rational numbers and } (a,b) \subseteq O\} \in \sigma(\mathcal{I})$$

countable union of set in $\mathcal{I}$

Hence $O \subseteq \sigma(\mathcal{I})$ and thus $\sigma(\mathcal{O}) = \sigma(\mathcal{I})$

a) The Borel σ-algebra is generated by the closed sets.

$\mathcal{O}$ = all the open sets $\qquad \sigma(\mathcal{O}) = \sigma(\mathcal{F})$

$\mathcal{F}$ = all closed sets

Assume that $O \in \mathcal{O}$. Then $O^c \in \mathcal{F} \subseteq \sigma(\mathcal{F})$.

Thus $O = (O^c)^c \in \sigma(\mathcal{F})$. This means that $\sigma(\mathcal{O}) \subseteq \sigma(\mathcal{F})$

Assume next that $F \in \mathcal{F}$. Then $F^c \in \mathcal{O} \subseteq \sigma(\mathcal{O})$

Thus $F = (F^c)^c \in \sigma(\mathcal{O})$

Problem 1.6:

$\mathcal{G} = \{ A \subseteq \mathbb{R} : A \text{ is countable or } A^c \text{ is countable} \}$

Prove that $\mathcal{G}$ is a σ-algebra.

(i) $\emptyset \in \mathcal{G}$ because $\emptyset$ is countable.

(ii) Assume $C \in \mathcal{G}$. Two cases:

a) C is countable $\Rightarrow$ $C^c \in \mathcal{G}$ because $(C^c)^c = C$ is countable.

b) C^c is countable $\Rightarrow$ $C^c \in \mathcal{G}$ because it is countable.

(iii) Assume that $C_n \in \mathcal{G}$ for all n. We need to prove that $\bigcup_{n\in\mathbb{N}} C_n \in \mathcal{G}$. If at least one C_n has a countable complement, then $\bigcup C_n \supseteq C_n$ also has a countable complement and hence $\bigcup_{n\in\mathbb{N}} C_n \in \mathcal{G}$.

But what if all the C_n's are countable. But the countable union of countable sets is countable and hence

$$\bigcup C_n \in \mathcal{G}$$

$$C_1 = \{c_{11}, c_{12}, c_{13}, c_{14}, \ldots\}$$

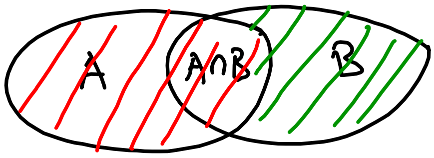
$$C_2 = \{c_{21}, c_{22}, c_{23}, c_{24}, \ldots\}$$

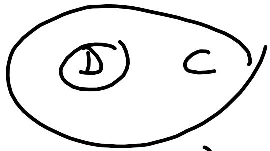
$$C_3 = \{c_{31}, c_{32}, c_{33}, c_{34}, \ldots\}$$

$$C_4 = \{c_{41}, c_{42}, c_{43}, c_{44}\}$$

1.22: Prove that $P(A \cup B) + P(A \cap B) = P(A) + P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$





$P(C \setminus D)$

$= P(C) - P(D)$

Proof: $P(A \cup B) = P(A) + P(B \setminus (A \cap B))$

$= P(A) + P(B) - P(A \cap B)$

1.23: Prove that: $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
$- P(A \cap B) - P(A \cap C) - P(B \cap C)$
$+ P(A \cap B \cap C)$

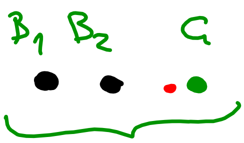
Proof: $P((A \cup B) \cup C) \overset{1.22}{=} P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

1.16:



G, B_1, B_2

G, B_2, B_1