STK-MAT3710/4710 Formula sheet

Families of sets

σ-algebra $\mathcal{F}$:

(i) $\emptyset \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$

(iii) $A_n \in \mathcal{F}$ for all $n \Longrightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$

Monotone class $\mathcal{M}$:

(i) $A_n \in \mathcal{M}$ increasing $\Longrightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$

(ii) $A_n \in \mathcal{M}$ decreasing $\Longrightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{M}$

Monotone Class Theorem: If $\mathcal{A}$ is an algebra, then $\sigma(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

Independence

Family $\{A_i\}_{i\in I}$ of sets: $P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$ for all finite subsets $\{i_1, i_2, \ldots, i_k\}$ of I

Family $\{X_i\}_{i\in I}$ of random variables: $[X_{i_1} \leq x_1], [X_{i_2} \leq x_2], \ldots, [X_{i_k} \leq x_k]$ independent for all finite subsets $\{i_1, i_2, \ldots, i_k\}$ of I and all $x_1, x_2, \ldots, x_k \in \mathbb{R}$.

Distributions

Distribution function F_X: $F_X(x) = P[X \leq x]$
Density function f_X: $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$
Distribution μ_X: $\mu_X(B) = P[X \in B]$
Lebesgue-Stieltjes integral: $\int_{-\infty}^{\infty} f(x)\,dF_X = \int_{-\infty}^{\infty} f(x)\,d\mu_X = E[f(X)]$
Characteristic function ϕ_X: $\phi_X(t) = E[e^{itX}]$
Taylor expansion: If $E[|X|^n] < \infty$, then $\phi^{(k)}(0) = i^k E[X^k]$ for $k \leq n$, and $\phi_X(t) = \sum_{k=0}^{n} \frac{1}{k!} E[X^k](it)^k + o(t^n)$.

Gaussian distribution $N(\mu, \sigma^2)$: Density function: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, characteristic function: $\phi(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$

Lévy's Inversion Theorem: $\bar{F}(b) - \bar{F}(a) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-2\pi it} \phi(t) e^{-\frac{\epsilon^2 t^2}{2}}\,dt$
Lévy's Continuity Theorem: If $\phi_{X_n}(t) \to \phi(t)$ and ϕ is continuous at 0, then X_n converges in distribution to a random variable X with $\phi_X = \phi$.

Modes of convergence

Convergence a.s.: $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all ω in a set of probability 1.
Convergence in probability: $\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0$ for all $\epsilon > 0$.
Convergence in expectation: $\lim_{n\to\infty} E[|X_n - X|] = 0$.
Convergence in distribution: $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all continuity points x of F_X. Equivalently: $E[f(X_n)] \to E[f(X)]$ for all bounded, continuous f (this is also called *weak convergence*).
Relationships: If $\{X_n\}$ converges to X a.s. or in expectation, then $\{X_n\}$ converges to X in probability. If $\{X_n\}$ converges to X a.s. or in probability, then $\{X_n\}$ converges to X in distribution. If $\{X_n\}$ converges to X in probability, there is a subsequence $\{X_{n_k}\}$ that converges to X a.s.

Convergence Theorems

Monotone Convergence Theorem: If $X_n \geq 0$ and $X_n \uparrow X$ a.s., then $E[X] = \lim_{n\to\infty} E[X_n]$.
Fatou's Lemma: If $X_n \geq 0$, then $\liminf_{n\to\infty} E[X_n] \geq E[\liminf_{n\to\infty} X_n]$.
Dominated Convergence Theorem: If $|X_n| \leq Y$ for an integrable r.v. Y, and $X_n \to X$ a.s. or in probability, then $E[X] = \lim_{n\to\infty} E[X_n]$.

Limit theorems

Below $S_n = X_1 + X_2 + \ldots + X_n$.
Weak law of large numbers: $\{X_n\}$ a sequence of independent random variables with $E[X_j] = 0$ and $E[X_j^2] \leq \sigma^2$. Then $\frac{S_n}{n} \to 0$ in probability.
Strong law of large numbers: $\{X_n\}$ a sequence of independent random variables with $E[X_j] = 0$ and $E[X_j^4] \leq M$. Then $\frac{S_n}{n} \to 0$ a.s.
Central limit theorem (i.i.d. version): $\{X_n\}$ a sequence of independent, identically distributed random variables with $E[X_j] = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Then $\frac{S_n - \mu n}{\sigma\sqrt{n}} \to N(0,1)$ in distribution.
Central limit theorem (Lyapounov version): $\{X_n\}$ a sequence of independent random variables with $E[X_j] = 0$ and $E[X_j^2] = \sigma_j^2$. Put $s_n^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$. Assume that $\gamma_j = E[|X_j|^3] < \infty$ and that $\frac{\sum_{j=1}^n \gamma_j}{s_n^3} \to 0$. Then $\frac{S_n}{s_n} \to N(0,1)$ in distribution.

Inequalities

Chebyshev's Inequality: For $\lambda > 0$: $P[|X| \geq \lambda] \leq \frac{1}{\lambda^p} E[|X|^p]$
Schwarz's Inequality: $E[|XY|] \leq (E[X^2])^{\frac{1}{2}} (E[Y^2])^{\frac{1}{2}}$
Lyapounov's Inequalities: For $1 \leq p < q$:

(i) $E[|X|]^q \leq E[|X|^q]$ (ii) $E[|X|^p]^{\frac{1}{p}} \leq E[|X|^q]^{\frac{1}{q}}$

Jensen's Inequality: For convex ϕ: $\phi(E[X|\mathcal{G}]) \leq E[\phi(X)|\mathcal{G}]$.

$\limsup$ and $\liminf$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k \qquad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k \qquad \limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

Tail Events

Borel-Cantelli's Lemma:

(i) If $\sum_{n=1}^\infty P(B_n) < \infty$, then $P[\limsup_{n\to\infty} B_n] = 0$.

(ii) If the B_n's are independent and $\sum_{n=1}^\infty P(B_n) = \infty$, then $P[\limsup_{n\to\infty} B_n] = 1$.

Borel/Kolmogorov's 0-1-Law: If the X_n's are independent and C is a tail event, then $P(C)$ is either 0 or 1.

Conditional expectation

Definition: $Z = E[X|\mathcal{G}]$ iff Z is $\mathcal{G}$-measurable and $\int_\Lambda Z\, dP = \int_\Lambda X\, dP$ for all $\Lambda \in \mathcal{G}$.
Tower property: If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $E\big[E[X|\mathcal{G}_2]|\mathcal{G}_1\big] = E[X|\mathcal{G}_1]$

Stopping times and martingales

Stopping time: $[T \le n] \in \mathcal{F}_n$ for all n (equivalently: $[T = n] \in \mathcal{F}_n$ for all n)
σ-algebra $\mathcal{F}_T$: $\mathcal{F}_T = \{\Lambda \in \mathcal{F} : \Lambda \cap [T \le n] \in \mathcal{F}_n \text{ for all } n\}$
Submartingale property: $E[X_t|\mathcal{F}_s] \ge X_s$ for $s < t$
Supermartingale property: $E[X_t|\mathcal{F}_s] \le X_s$ for $s < t$
Martingale property: $E[X_t|\mathcal{F}_s] = X_s$ for $s < t$.
Martingale Maximal Inequality: For a positive submartingale X_n:
$\lambda P[\max_{n \le N} X_n \ge \lambda] \le E[X_N]$
Martingale Convergence Theorem: If $\{X_n\}_{n\in\mathbb{N}}$ is a submartingale and $\sup_{n\in\mathbb{N}} E[X_n] < \infty$, then $X_n \to X_\infty$ a.s. where X_∞ is integrable.

Uniform integrability

Definition: For all $\epsilon > 0$ there is an N such that $\int_{[|X_\alpha| \ge N]} |X_\alpha|\, dP < \epsilon$ for all $\alpha \in I$.
Alternative definition: $\{E[|X_\alpha|]\}_{\alpha \in I}$ is bounded and $\lim_{P(\Lambda)\to 0} \int_\Lambda |X_\alpha|\, dP = 0$ uniformly in α.
Main Theorem: The following are equivalent when $X_n \to X$ in probability:

(i) $\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

(ii) $E[|X_n - X|] \to 0$.

(iii) $E[|X_n|] \to E[|X|]$.

Series and such

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + e^c \frac{x^{n+1}}{(n+1)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + (-1)^{n+1} \frac{\cos c}{(2n+3)!} x^{2n+3}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{\cos c}{(2n+2)!} x^{2n+2}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \frac{1}{(1+c)^{n+1}(n+1)} x^{n+1}$$

Lemma: If $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \left(1 + \frac{z_n}{n}\right)^n = e^z$.