UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	STK-MAT3710/4710 — Probability Theory.
Day of examination:	Wednesday, December 11th, 2019.
Examination hours:	14.30 – 18.30.
This problem set consists of 2 pages.	
Appendices:	Formula sheet.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1, 2a, 2b etc.) count equally. If there is a problem you can not solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (10 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be independent and identically distributed random variables taking values in the set $\{1,2,3,4,5,6\}$. Assume that $P([X_n = i]) = \frac{1}{6}$ for $i = 1,2,3,4,5,6$. Show that for almost all ω, the sequence $X_1(\omega), X_2(\omega), X_3(\omega), \ldots$ contains infinitely many occurences of 17 consecutive 6's.

Problem 2

A distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x+1) & \text{for } -1 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

a) (10 points) Find a real expression for the characteristic function of F.

b) (10 points) Assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with distribution function F. Find the characteristic function of

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\ .$$

c) (10 points) Use the result in b) to "show directly" that S_n converges in distribution to a normal distribution ("show directly" means that you are not allowed to use a version of the Central Limit Theorem).

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Problem 3

In this problem, $(\Omega, \mathcal{F}, P)$ is a probability space; $\mathbb{N}$ is the timeline; and $\{M_n\}_{n\in\mathbb{N}}$ is a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$. We assume that all M_n have finite second moments, and for each $n \in \mathbb{N}$, we let $\Delta M_n = M_{n+1} - M_n$ be the forward increment of M.

a) (10 points) Show that if $n \leq m$, then $E[\Delta M_m|\mathcal{F}_n] = 0$ and $E[\Delta M_m M_n|\mathcal{F}_n] = 0$.

b) (10 points) Show that if $n < m$, then $E[\Delta M_m \Delta M_n|\mathcal{F}_n] = 0$.

c) (10 points) Show that if $n < m$, then

$$E[(M_m - M_n)^2|\mathcal{F}_n] = \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

Problem 4

Recall that a function $F\colon \mathbb{R} \to \mathbb{R}$ is a distribution function if

(i) F is right continuous and increasing

(ii) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

a) (10 points) Show that if F is a distribution function and Y is a random variable, then

$$K(x) = E[F(x - Y)]$$

is also a distribution function. (To save time, you need only check one of the two conditions $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.)

In the rest of the problem, $X, Y\colon \Omega \to \mathbb{R}$ are two independent random variables with distribution functions F and G, respectively. Our aim is to show that the distribution function H of $X + Y$ is given by

$$H(x) = E[F(x - Y)]\ . \tag{1}$$

b) (10 points) Assume first that Y is of the form $Y = \sum_{n=1}^{\infty} a_n \mathbf{1}_{A_n}$, where the a_n's are distinct real numbers and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of Ω; i.e. the sets are mutually disjoint and their union is all of Ω. Show that (1) holds in this case. Specify where you use the independence of X and Y.

c) (10 points) If Y is a general random variable, let

$$\underline{Y}_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}$$

be the usual lower approximation of Y. Show that

$$E[F(x - Y)] = \lim_{n\to\infty} E[F(x - \underline{Y}_n)]\ .$$

d) (10 points) Show that equation (1) holds for all random variables Y that are independent of X. Conclude that

$$H(x) = \int_{-\infty}^{\infty} F(x - y)\, dG(y)$$

THE END