Exam in STK-MAT3710/4710, Fall 2019. Solutions

Problem 1

Cut $\mathbb{N}$ into sequences of length 17:

$I_0 = \{1, 2, \dots, 17\}, I_1 = \{18, 19, \dots, 34\}, \dots, I_k = \{17k+1, 17k+2, \dots, 18k\}, \dots$.

The sets

$$A_k = \{\omega : X_j(\omega) = 6 \text{ for all } j \in I_k\}$$

are independent and have probability $P(A_k) = \frac{1}{6^{17}} > 0$. Hence $\sum_{k=0}^{\infty} P(A_k) = \infty$. According to the Converse Borel-Cantelli Lemma, $\limsup A_k$ has probability 1, and each ω in $\limsup A_k$ clearly contains infinitely many occurences of 17 consecutive 6's.

Problem 2

a) Differentiating we see that the distribution has the density

$$f(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2} & \text{for} -1 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases}$$

The characteristic function is

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \frac{1}{2}\int_{-1}^{1} e^{itx}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_{x=-1}^{x=1} = \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}$$

b) We have

$$\phi_{S_n}(t) = E\left[e^{itS_n}\right] = E\left[e^{i\frac{t}{\sqrt{n}}X_1} e^{i\frac{t}{\sqrt{n}}X_2} \cdot \ldots \cdot e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1}\right] E\left[e^{i\frac{t}{\sqrt{n}}X_2}\right] \cdot \ldots \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n}\right] = \left(\phi(\frac{t}{\sqrt{n}})\right)^n = \left(\frac{\sin\frac{t}{\sqrt{n}}}{\frac{t}{\sqrt{n}}}\right)^n$$

c) The Taylor expansion of the sine function is

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

and hence

$$\frac{\sin\frac{t}{\sqrt{n}}}{\frac{t}{\sqrt{n}}} = 1 - \frac{t^2}{6n} + o\left(\frac{t^2}{n}\right)$$

Consequently (e.g. by Lemma 6.34)

$$\phi_{S_n}(t) = \left(1 - \frac{t^2}{6n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-\frac{t^2}{6}}.$$

As $e^{-\frac{t^2}{6}}$ is the characteristic function of a normal distribution with mean 0 and variance $\sigma^2 = \frac{1}{3}$, the result follows from Lévy's Continuity Theorem.

Problem 3

a) Note first that

$$E[\Delta M_m|\mathcal{F}_n] = E[M_{m+1}|\mathcal{F}_n] - E[M_m|\mathcal{F}_n] = M_n - M_n = 0$$

by the martingale property.

For the second part, observe that since M_n is $\mathcal{F}_n$-measurable, we have

$$E[\Delta M_m M_n|\mathcal{F}_n] = M_n E[\Delta M_m|\mathcal{F}_n] = 0$$

where the last step uses what we just proved above.

b) As ΔM_n is $\mathcal{F}_m$-measurable, we have by the tower property

$$E[\Delta M_m \Delta M_n|\mathcal{F}_n] = E\big[E[\Delta M_n \Delta M_m|\mathcal{F}_m]|\mathcal{F}_n\big]$$

$$= E\big[\Delta M_n E[\Delta M_m|\mathcal{F}_m]|\mathcal{F}_n\big] = E\big[\Delta M_n \cdot 0|\mathcal{F}_n\big] = 0$$

c) Since $M_m - M_n = \sum_{k=n}^{m-1} \Delta M_k$, we have

$$E\left[(M_m - M_n)^2|\mathcal{F}_n\right] = E\left[\left(\sum_{k=n}^{m-1} \Delta M_k\right)^2 \Bigg|\mathcal{F}_n\right] = \sum_{i,j=n}^{m-1} E\left[\Delta M_i \Delta M_j|\mathcal{F}_n\right]$$

$$= 2\sum_{n \le i < j \le m-1} E[\Delta M_i \Delta M_j|\mathcal{F}_n] + \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

All terms in the first sum are zero since by b) and the tower property, we have:

$$E[\Delta M_i \Delta M_j|\mathcal{F}_n] = E\left[E[\Delta M_i \Delta M_j|\mathcal{F}_i]|\mathcal{F}_n\right] = E[0|\mathcal{F}_n] = 0$$

Hence

$$E[(M_m^2 - M_n)^2|\mathcal{F}_n] = \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

Problem 4

a) Assume $x_1 < x_2$. As F is increasing, we have

$$K(x_1) = E[F(x_1 - Y)] \le E[F(x_2 - Y)] = K(x_2)$$

which shows that K is increasing. To prove right continuity, note that if $\{x_n\}$ is a sequence decreasing to x, the Bounded Convergence Theorem tells us that (note that the integrand is bounded by 1):

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[F(x - Y)] = K(x)$$

since F is right continuous. It follows that K is right continuous.

To check the limit conditions, first note that if $x \to -\infty$, then $x - Y \to -\infty$, and hence $\lim_{x\to-\infty} F(x-Y) = 0$. Applying the Bounded Convergence Theorem again, we see that if $x_n \to -\infty$, then

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[0] = 0$$

The other limit is similar: If $x_n \to \infty$, then

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[1] = 1$$

as $\lim_{x\to\infty} F(x) = 1$.

b) We have

$$H(x) = P[X + Y \leq x] = P[X + \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \leq x] = \sum_{n\in\mathbb{N}} P([X \leq x - a_n] \cap A_n)$$

$$= \sum_{n\in\mathbb{N}} P([X \leq x - a_n] \cap [Y = a_n]) = \sum_{n\in\mathbb{N}} P[X \leq x - a_n] P[Y = a_n]$$

$$= \sum_{n\in\mathbb{N}} P[X \leq x - a_n] P[A_n] = \sum_{n\in\mathbb{N}} F(x - a_n) P[A_n] = E[F(x - Y)]$$

where we have used the independence to get the equality in the middle of the second line.

c) As $\underline{Y}_n$ increases to Y, we see that $x - \underline{Y}_n$ approaches $x - Y$ from the right, and since F is right continuous, it follows that $F(x - Y_n) \to F(x - Y)$. Using the Bounded Convergence Theorem again, we get

$$\lim_{n\to\infty} E[F(x - \underline{Y}_n)] = E[\lim_{n\to\infty} F(x - \underline{Y}_n)] = E[F(x - Y)] \ .$$

d) If H_n is the distribution function of $X + Y_n$, we know from b) that

$$H_n(x) = E[F(x - Y_n)] \ ,$$

and we have already seen that the right hand side converges pointwise to $K(x) = E[F(x - Y)]$ which is a distribution function by a). On the other hand, H_n is the distribution function of $X + \underline{Y}_n$ and since $X + \underline{Y}_n$ converges pointwise (and hence in distribution) to $X + Y$, the distribution functions H_n converge to the distribution function H of $X + Y$ at all continuity points of H. This means that the two distribution functions H and K coincide at all continuity points of H, and hence they have to be equal for all x.

The last formula,

$$H(x) = \int_{-\infty,\infty} F(x-y)\, dG(y)$$

follows immediately from $H(x) = E[F(x-Y)]$ as we have $E[f(Y)] = \int_{-\infty}^{\infty} f(y)\, dG(y)$ for all Borel functions f.