STK-MAT3710: Solution to Exam 2020

Problem 1 a) We have

$$\phi_Y(t) = E[e^{itY}] = \sum_{n=0}^{\infty} e^{itn} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^{it})^n}{n!} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

where we have used the Taylor series for the exponential function.

b) Note first that since the series

$$\sum_{n=0} n \frac{\lambda^n}{n!} e^{-\lambda} = \sum_{n=1} \frac{\lambda^n}{(n-1)!} e^{-\lambda}$$

converges, Y is integrable, and hence ϕ_Y is differentiable with $\phi_Y'(0) = iE(Y)$. Differentiating we get

$$\phi_Y'(t) = e^{\lambda(e^{it}-1)} \lambda i e^{it} = i\lambda e^{it} e^{\lambda(e^{it}-1)},$$

which means that $\phi_Y'(0) = i\lambda$. Thus

$$E[Y] = \frac{\phi_Y'(0)}{i} = \frac{i\lambda}{i} = \lambda.$$

c) We have

$$\phi_X(t) = E[e^{itX}] = e^{it0}(1 - \frac{\lambda}{n}) + e^{it1}\frac{\lambda}{n} = 1 + \left(e^{it} - 1\right)\frac{\lambda}{n}.$$

d) By independence, we have

$$\begin{aligned}\phi_{S_n} &= E[e^{itS_n}] = E[e^{it(X_n^{(1)} + \cdots + X_n^{(n)})}] = E[e^{itX_n^{(1)}}] \cdot \ldots \cdot E[e^{itX_n^{(n)}}] \\ &= \phi_X(t)^n = \left(1 + \left(e^{it} - 1\right)\frac{\lambda}{n}\right)^n.\end{aligned}$$

e) By the definition of e (or a formula on the formula sheet),

$$\lim_{n\to\infty} \phi_{S_n}(t) = \lim_{n\to\infty} \left(1 + \left(e^{it} - 1\right)\frac{\lambda}{n}\right)^n = e^{\lambda(e^{it}-1)}$$

which is the characteristic function of Y, and hence S_n converges to Y in distribution by Lévy's Continuity Theorem.

Problem 2 a) M_n is obviously $\mathcal{F}_n$-measurable and integrable. Putting $\Delta M_n = M_{n+1} - M_n = X_{n+1}$ and using that X_{n+1} is independent of $\mathcal{F}_n$, we have

$$E[\Delta M_n \,|\, \mathcal{F}_n] = E[X_{n+1} \,|\, \mathcal{F}_n] = E[X_{n+1}] = ap + b(1-p) = (a-b)p + b$$

This quantity is 0 when $p = \frac{-b}{a-b}$ (which is between 0 and 1 since a is positive and b is negative), it is positive when $p \geq \frac{-b}{a-b}$, and it is negative when $p \leq \frac{-b}{a-b}$. Hence M is a martingale when $p = \frac{-b}{a-b}$, a submartingale when $p \geq \frac{-b}{a-b}$, and a supermartingale when $p \leq \frac{-b}{a-b}$.

b) Observe first that with $a = 1$, $b = -1$, the critical value in a) becomes $p = \frac{-b}{a-b} = \frac{1}{2}$. Note also that since T is a first hitting time, it is a stopping time. Let $Y_n = M_{n \wedge T}$. Then Y is bounded and $Y_T = M_T$. Moreover, Y is a martingale/submartingale/supermartingale iff M is one. Applying Theorem 9.11 (optional stopping for bounded processes) to the stopping times 1 and T, we get $E[Y_T] = E[Y_1]$ if Y is a martingale, $E[Y_T] \geq E[Y_1]$ if Y is a submartingale, and $E[Y_T] \leq E[Y_1]$ if Y is a supermartingale. As $E[Y_1] = E[M_1]$ is zero, greater than zero, or less than 0 according to whether $p = \frac{1}{2}$, $p > \frac{1}{2}$, or $p < \frac{1}{2}$, we see that $E[M_T] = E[M_1] = 0$ if $p = \frac{1}{2}$, $E[M_T] \geq E[M_1] > 0$ if $p > \frac{1}{2}$, and $E[M_T] \leq E[M_1] < 0$ if $p < \frac{1}{2}$.

Problem 3 a) We have

$$E[Y_N] = E[X_1 + X_2 + \cdots + X_N] = E[X_1] + E[X_2] + \cdots + E[X_N] = 0.$$

As the X_n are independent,

$$\mathrm{var}(Y_n) = \mathrm{var}(X_1 + X_2 + \cdots + X_N) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_N) = N.$$

b) We have

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \leq \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\,dx$$

since $e^{-\frac{x^2}{2}} \leq \frac{x}{a} e^{-\frac{x^2}{2}}$ on the interval $[a, \infty)$ that we are integrating over. Hence using the substitution $u = \frac{x^2}{2}$, $du = x\,dx$, we get

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \leq \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\,dx = \int_{\frac{a^2}{2}}^\infty \frac{1}{a} e^{-u}\,du = \left[-\frac{1}{a} e^{-u}\right]_{\frac{a^2}{2}}^\infty = \frac{1}{a} e^{-\frac{a^2}{2}}.$$

c) Since Y_N is gaussian with mean 0 and variance N, we have

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] = \int_{\sqrt{N^{1+\epsilon}}}^\infty \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2N}}\,du.$$

Substituting $x = \frac{u}{\sqrt{N}}$, $dx = \frac{du}{\sqrt{N}}$ and using the inequality above, we get

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] = \int_{N^{\frac{\epsilon}{2}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,du \leq \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi} N^{\frac{\epsilon}{2}}} \leq \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi}}.$$

d) By the first half of Borel-Cantelli's lemma,

$$P\left[Y_N > \sqrt{N^{1+\epsilon}} \text{ for infinitely many } N\right] = 0$$

if

$$\sum_{N=1}^\infty P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] < \infty.$$

By c)

$$\sum_{N=1}^\infty P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] \leq \frac{1}{\sqrt{2\pi}} \sum_{N=1}^\infty e^{-\frac{N^\epsilon}{2}},$$

and hence it suffices to show that the series $\sum_{N=1}^{\infty} e^{-\frac{N^\epsilon}{2}}$ converges. Using the hint with $p = 2$, we see that

$$\lim_{N\to\infty} \frac{e^{-\frac{N^\epsilon}{2}}}{\frac{1}{N^2}} = \lim_{N\to\infty} \frac{N^2}{e^{\frac{N^\epsilon}{2}}} = 0.$$

Since $\sum_{N=1}^{\infty} \frac{1}{N^2}$ converges, this means that $\sum_{N=1}^{\infty} e^{-\frac{N^\epsilon}{2}}$ converges by the Limit Comparison Test, and we are done.

(To see that the hint is correct, choose $m \in \mathbb{N}$ so large that $m\epsilon > p$ and note that by Taylor's Formula

$$e^{\frac{N^\epsilon}{2}} > \sum_{k=0}^{m} \frac{(\frac{N^\epsilon}{2})^k}{k!} > \frac{N^{m\epsilon}}{2^m m!}.$$

Hence

$$\lim_{N\to\infty} \frac{N^p}{e^{\frac{N^\epsilon}{2}}} \leq \lim_{N\to\infty} \frac{N^p}{\frac{N^{m\epsilon}}{2^m m!}} = 0$$

as $m\epsilon > p$.)