UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3710/4710 — Probability Theory.

Day of examination: Monday, December 5th, 2022.

Examination hours: 15.00 – 19.00.

This problem set consists of 3 pages.

Appendices: Formula sheet.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b, 2 etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

We use $\mathbb{N}_0$ to denote the natural numbers with 0 included, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

Problem 1 (30 points)

a) Let a be a real number and assume that Y_a is a random variable such that

$$Y_a = \begin{cases} 1 + a & \text{with probability } \frac{1}{2} \\ -1 + a & \text{with probability } \frac{1}{2} \end{cases}$$

Show that the characteristic function of Y_a is $\phi_a(t) = e^{ita} \cos t$.

b) For $n \in \mathbb{N}$, let $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$ be n independent copies of $Y_{\frac{1}{\sqrt{n}}}$, and put

$$S_n = \frac{X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)}}{\sqrt{n}}$$

Find the characteristic function ϕ_{S_n} of S_n.

c) Show that $\{S_n\}$ converges in distribution to a normal distribution. What is its mean and variance?

Problem 2 (10 points) A radio station is running a lottery every day. The first day the chance of winning is $\frac{1}{100}$, the second day it is $\frac{1}{101}$, the third day $\frac{1}{102}$ and so on. What is the probability of winning infinitely many times if you live forever and play the lottery every day?

(Continued on page 2.)

Problem 3 (10 points) Assume that X and Y are two integrable random variables. Show that the family of all sets Λ such that

$$\int_\Lambda X\,dP = \int_\Lambda Y\,dP$$

is a monotone class.

Problem 4 (10 points) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of independent events. Let

$$P_n = P(A_1) + P(A_2) + \cdots + P(A_n)$$

and let $X_n(\omega)$ be the number of sets among $A_1, A_2, \ldots, A_n$ that ω belongs to, i.e., $X_n(\omega) = |\{i \le n \,|\, \omega \in A_i\}|$. Show that the sequence $\left\{\frac{X_n - P_n}{n}\right\}$ converges to 0 a.s.

Problem 5 (50 points)

We assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative, integrable random variables with finite mean $E[X_n] = \mu_n > 0$. We also assume that Y is a random variable which takes values in $\mathbb{N}_0$ and is independent of $\{X_n\}_{n\in\mathbb{N}}$. Define a random variable Z by

$$Z(\omega) = \begin{cases} X_1(\omega) + X_2(\omega) + \cdots + X_{Y(\omega)}(\omega) & \text{if } Y(\omega) > 0 \\ \\ 0 & \text{if } Y(\omega) = 0 \end{cases}.$$

a) Show that

$$Z(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\omega)(X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega))$$

and that

$$E[Z] = \sum_{n=1}^{\infty} (\mu_1 + \mu_2 + \cdots + \mu_n) P[Y = n]$$

b) Assume that $\mathcal{G}$ is a σ-algebra such that Y is $\mathcal{G}$-measurable and the $\{X_n\}$'s are independent of $\mathcal{G}$. Show that if Z is integrable, then

$$E[Z|\mathcal{G}](\omega) = \sum_{n=1}^{\infty} (\mu_1 + \cdots + \mu_n) \mathbf{1}_{[Y=n]}(\omega)$$

In the rest of the problem, we assume that X is a random variable with mean $\mu > 0$ taking values in $\mathbb{N}_0$. We shall study a probabilistic model for how a population of animals develops from generation to generation. If generation k consists of $Z_k(\omega) \in \mathbb{N}_0$ individuals, we assume that each of these gives rise to a random number of offsprings distributed according to X. We also assume that the number of offsprings is independent from individual to individual and from generation to generation.

To model this mathematically, we assume that $\{X_i^{(k)}\}_{k,i\in\mathbb{N}}$ are independent copies of X. We let $Z(0) \in \mathbb{N}$ and define inductively

$$Z_{k+1}(\omega) = \begin{cases} X_1^{(k+1)}(\omega) + X_2^{(k+1)}(\omega) + \cdots + X_{Z_k(\omega)}^{(k+1)}(\omega) & \text{if } Z_k(\omega) > 0 \\ \\ 0 & \text{if } Z_k(\omega) = 0 \end{cases}$$

We also assume that X is chosen such that Z_k is integrable for all k.

(Continued on page 3.)

c) Let $\mathcal{F}_k$ be the σ-algebra generated by all $\{X_i^{(j)}\}$ with $j \leq k$. Show that

$$E[Z_{k+1}|\mathcal{F}_k] = \mu Z_k$$

For which values of μ is $\{Z_k\}$ an $\{\mathcal{F}_k\}$-submartingale, an $\{\mathcal{F}_k\}$-martingale, or an $\{\mathcal{F}_k\}$-supermartingale, respectively?

d) Show that $Y_k = \frac{Z_k}{\mu^k}$ is an $\{\mathcal{F}_k\}$-martingale.

One can prove that if X has finite variance σ^2, then there is a $K \in \mathbb{R}$ such that $E[Y_k^2] \leq K$ for all k (you can use this without proof), and hence $\{Y_k\}$ is uniformly integrable.

e) Assume that X has finite variance and that $\mu > 1$. Explain that $\{Y_k\}$ converges almost surely and in L^1 to an integrable random variable Y_∞. Show that there must be a set of positive probability where Y_∞ is strictly positive, and conclude that Z_k goes to infinity with positive probability.

GOOD LUCK