STK-MAT3710/4710: Solution to Exam 2022

Problem 1 a) We have

$$\phi_a(t) = E[e^{itY_a}] = e^{it(1+a)} \cdot \frac{1}{2} + e^{it(-1+a)} \cdot \frac{1}{2}$$

$$= e^{iat}\frac{e^{it} + e^{-it}}{2} = e^{ita}\cos t$$

b) Using the independence of $X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}$, we get

$$\phi_{S_n}(t) = E[e^{itS_n}] = E\left[e^{it\frac{X_1^{(n)}+X_2^{(n)}+\cdots+X_n^{(n)}}{\sqrt{n}}}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1^{(n)}}\right] \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_2^{(n)}}\right] \cdot \ \ldots \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n^{(n)}}\right]$$

$$= \left[\phi_{\frac{1}{\sqrt{n}}}\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[e^{i\frac{t}{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}}\cos\frac{t}{\sqrt{n}}\right]^n = e^{it}\left[\cos\frac{t}{\sqrt{n}}\right]^n$$

c) Using the Taylor expansion $\cos x = 1 - \frac{x^2}{2} + o(x^2)$, we get

$$\lim_{n\to\infty}\left[\cos\frac{t}{\sqrt{n}}\right]^n = \lim_{n\to\infty}\left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n = e^{-\frac{t^2}{2}}$$

by Lemma 6.34. This means that

$$\lim_{n\to\infty}\phi_{S_n}(t) = e^{it-\frac{t^2}{2}}$$

which is the characteristic function of an $\mathcal{N}(1,1)$ random variable. By Lévy's Continuity Theorem, S_n converges in distribution to a normal distribution with mean 1 and variance 1.

Problem 2 Let A_n be the event that you win on day n. Then the A_n's are independent (we must assume) and $P(A_n) = \frac{1}{99+n}$. The series

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty}\frac{1}{99+n}$$

diverges (it is the tail of the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$), and thus the probability of winning infinitely many times is 1 by the Borel-Cantelli Lemma.

Problem 3 Let $\mathcal{M} = \{\Lambda : \int_\Lambda X\,dP = \int_\Lambda Y\,dP\}$; we must prove that $\mathcal{M}$ is closed under increasing unions and decreasing intersections.

Assume first that $\{\Lambda_n\}$ is an increasing sequence of sets in $\mathcal{M}$ with union Λ. Since the sequence $\{\mathbf{1}_{\Lambda_n}X\}$ is dominated by the integrable function $|X|$, the Dominated Convergence Theorem tells us that

$$\int_\Lambda X\,dP = \int \mathbf{1}_\Lambda X\,dP = \int \lim_{n\to\infty}\mathbf{1}_{\Lambda_n}X\,dP \overset{DCT}{=} \lim_{n\to\infty}\int \mathbf{1}_{\Lambda_n}X\,dP = \lim_{n\to\infty}\int_{\Lambda_n} X\,dP$$

and similarly,

$$\int_\Lambda Y\,dP = \lim_{n\to\infty}\int_{\Lambda_n} Y\,dP.$$

This shows that $\int_\Lambda X\,dP = \int_\Lambda Y\,dP$, and hence $\Lambda \in \mathcal{M}$.

Assume now that $\{\Lambda_n\}$ is a decreasing sequence of sets in $\mathcal{M}$ with intersection Λ. As above, the sequence $\{\mathbf{1}_{\Lambda_n}X\}$ is dominated by the integrable function $|X|$, and the Dominated Convergence Theorem gives

$$\int_\Lambda X\,dP = \int \mathbf{1}_\Lambda X\,dP = \int \lim_{n\to\infty}\mathbf{1}_{\Lambda_n}X\,dP \overset{DCT}{=} \lim_{n\to\infty}\int \mathbf{1}_{\Lambda_n}X\,dP = \lim_{n\to\infty}\int_{\Lambda_n} X\,dP$$

and similarly,

$$\int_\Lambda Y\,dP = \lim_{n\to\infty}\int_{\Lambda_n} Y\,dP.$$

This shows that $\int_\Lambda X\,dP = \int_\Lambda Y\,dP$, and hence $\Lambda \in \mathcal{M}$.

Problem 4 Let $Y_n = \mathbf{1}_{A_n}$. Then $E[Y_n] = P(A_n)$, and the random variables $Z_n = Y_n - P(A_n)$ have mean 0. Since the Z_n's are independent and obviously have bounded fourth moments, Cantelli's Strong Law of Large Numbers tells us that $\frac{Z_1+Z_2+\cdots+Z_n}{n}$ converges to 0 a.s. But $X_n = Y_1 + Y_2 + \ldots + Y_n$ and $P_n = P(A_1) + P(A_2) + \cdots + P(A_n)$, and thus

$$\frac{X_n - P_n}{n} = \frac{Z_1 + Z_2 + \cdots + Z_n}{n} \to 0 \qquad \text{a.s.}$$

Problem 5

a) To show that

$$Z(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\omega)(X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega)),$$

we first observe that if $Y(\omega) = 0$, then both sides are 0. Next we observe that if $Y(\omega) = n$ for $n > 0$, then both sides equal $X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega)$, and hence the expressions are equal for all ω.

Taking expectations on both sides of the equation above, we get

$$E[Z] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

Since all terms are positive, the Monotone Convergence Theorem gives

$$E[Z] = E\left[\lim_{m\to\infty}\sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

$$= \lim_{m\to\infty} E\left[\sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

Using the independence of Y and the X_i's, we see that this equals

$$\lim_{m\to\infty}\left[\sum_{n=1}^{m} E[\mathbf{1}_{[Y=n]}]E[X_1 + X_2 + \cdots + X_n]\right]$$

$$= \lim_{m\to\infty} \left[\sum_{n=1}^{m} P[Y=n](\mu_1+\mu_2+\cdots+\mu_n) \right]$$

$$= \sum_{n=1}^{\infty} P[Y=n](\mu_1+\mu_2+\cdots+\mu_n)$$

and hence

$$E[Z] = \sum_{n=1}^{\infty} (\mu_1+\mu_2+\cdots+\mu_n) P[Y=n]$$

b) Using the Monotone Convergence Theorem for conditional expextations, we get

$$E[Z|\mathcal{G}] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(X_1+X_2+\cdots+X_n)|\mathcal{G} \right]$$

$$= \lim_{m\to\infty} E\left[\sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1+X_2+\cdots+X_n)|\mathcal{G} \right]$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} E[\mathbf{1}_{[Y=n]}(X_1+X_2+\cdots+X_n)|\mathcal{G}]$$

Since Y is $\mathcal{G}$-measurable and the $\{X_n\}$'s are independent of $\mathcal{G}$, we have

$$E[\mathbf{1}_{[Y=n]}(X_1+X_2+\cdots+X_n)|\mathcal{G}] = \mathbf{1}_{[Y=n]} E[X_1+X_2+\cdots+X_n|\mathcal{G}]$$

$$= \mathbf{1}_{[Y=n]}(\mu_1+\mu_2+\cdots+\mu_n)$$

and hence

$$E[Z|\mathcal{G}] = \lim_{m\to\infty} \sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(\mu_1+\mu_2+\cdots+\mu_n)$$

$$= \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\mu_1+\mu_2+\cdots+\mu_n).$$

c) Since Z_k is $\mathcal{F}_k$-measurable by induction, and $X_i^{(k+1)}$ is independent of $\mathcal{F}_k$ by assumption, we get from b) that

$$E[Z_{k+1}|\mathcal{F}_k] = \sum_{n=1}^{\infty} \mathbf{1}_{[Z_k=n]}(\mu+\mu+\cdots+\mu) = \mu \sum_{n=1}^{\infty} n\mathbf{1}_{[Z_k=n]} = \mu Z_k.$$

Since $Z_k \geq 0$, we see that $E[Z_{k+1}|\mathcal{F}_k] \geq Z_k$ when $\mu > 1$, $E[Z_{k+1}|\mathcal{F}_k] = Z_k$ for $\mu = 1$, and $E[Z_{k+1}|\mathcal{F}_k] \leq Z_k$ when $\mu < 1$. This means that Z_k is a submartingale, a martingale, and a supermartingale according to whether $\mu \geq 1$, $\mu = 1$, or $\mu \leq 1$.

d) Using c), we see that

$$E[Y_{k+1}|\mathcal{F}_k] = \frac{1}{\mu^{k+1}} E[Z_{k+1}|\mathcal{F}_k] = \frac{\mu}{\mu^{k+1}} Z_k = Y_k$$

which shows that $Y_k = \frac{Z_k}{\mu^k}$ is a martingale.

e) We use Theorem 6.31: Since $\{Y_k\}$ is a uniformly intergrable martingale, Y_k converges a.s. and in L^1 to an integrable random variable Y_∞ such that the augmented process $Y_0, Y_1, \ldots, Y_\infty$ is also a martingale. Hence $E[Y_\infty] = E[Y_0] = E[Z_0] = Z_0 > 0$, which proves that Y_∞ is positive on a set Ω_0 of positive measure. Since $Z_k = \mu^k Y_k$, it follows that Z_k goes to infinity on Ω_0.