STK-MAT3710: Filtrations, stopping times, and intuition

The notions of filtrations and stopping times can be hard to grasp at first. In this note I'll try to make the intuition clearer by first looking at a particular filtration where things fit together nicely.

The setting is as follows: $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{X_n\}_{n\in\mathbb{N}}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$. As usual, we shall think of n as time. We want to capture the information we get from observing the process up to time n. If ω and ω' are two elements in Ω such that $X_i(\omega) = X_i(\omega')$ for all $i \leq n$, we clearly cannot distinguish ω from ω' based on our observations up to time n. Hence if we want to make a decision at time n based on our observations so far, that decision has to be the same in case ω as in case ω'. Let us make this a bit more formal.

Definition 1 *We say that* $\omega, \omega' \in \Omega$ *are* indistinguishable at time n, *and write* $\omega \sim \omega'$, *if* $X_i(\omega) = X_i(\omega')$ *for all* $i \leq n$. *We shall also write*

$$[\omega]_n = \{\omega' \in \Omega \,:\, \omega' \sim_n \omega\}$$

for the set of all ω' *that are indistinguishable from* ω *at time* n.

Take an event $A \in \mathcal{F}$. What should it mean that A is observable at time n? Well, it ought to mean that we can decide whether an $\omega \in \Omega$ belongs to A by looking at our observations $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$, i.e.:

Definition 2 *An event* $A \in \mathcal{F}$ *is* observable at time n *if whenever* $\omega \in A$, *then* $[\omega]_n \subseteq A$ *(hence if* ω *is in* A, *all* ω' *which are indistinguishable from* ω *at time* n *are also in* A*).*

Here comes the first connection to σ-algebras:

Proposition 3 *The events that are observable at time* n *form a* σ*-algebra that we shall denote by* $\mathcal{A}_n$.

Proof: We have to check the three conditions for σ-algebras:

(i) As $\emptyset$ doesn't have any element, it clearly satisfies the criterion for being in $\mathcal{A}_n$.

(ii) Assume $A \in \mathcal{A}_n$; we must prove that $A^c \in \mathcal{A}_n$. Assume not; then there must be a $\omega \in A^c$ such that $[\omega]_n \not\subseteq A^c$. Hence there is an $\omega' \in [\omega]_n$ such that $\omega' \in A$. Since $A \in \mathcal{A}_n$, this means that $[\omega'] \subseteq A$. But $\omega \in [\omega']_n$, and hence we have our contradiction as ω cannot both be in A^c and A.

(iii) Assume that $A_k \in \mathcal{A}_n$ for all $k \in \mathbb{N}$; we must show that $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}_n$: If $\omega \in \bigcup_{k\in\mathbb{N}} A_k$, then $\omega \in A_k$ for some $k \in \mathbb{N}$, and since $A_k \in \mathcal{A}_n$, this means that $[\omega]_n \subseteq A_k$. But then $[\omega]_n \subseteq \bigcup_{k\in\mathbb{N}} A_k$, which is what we had to prove. □

I leave to you to check that everything is in order with respect to measurability:

Proposition 4 *$\{\mathcal{A}_n\}$ is an increasing sequence of σ-algebras and X_n is $\mathcal{A}_n$-measurable for all n.*

Let us take next a look at stopping times. The idea is that a stopping time is a decision we can make based on the information we have available at the time. Hence if $T(\omega) = n$ and $\omega' \sim_n \omega$, we should also have $T(\omega') = n$ as we at time n cannot distinguish between ω and ω'. Let us turn this into a formal definition:

Definition 5 *A random variable $T\colon \Omega \to \mathbb{N} \cup \{\infty\}$ is a* stopping time *(with respect to $\{\mathcal{A}_n\}$) if whenever $T(\omega) = n$ and $\omega' \sim_n \omega$, then $T(\omega') = n$.*

The problem with this definition is that although it makes perfect sense for the filtration $\mathcal{A}_n$, it is hard to generalize to other filtrations. For that reason we want to reformulate it in a way that is easier to generalize.

Proposition 6 *A random variable $T\colon \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time if and only if $\{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$ for all n.*

Proof: Assume first that T is a stopping time according to the definition above. For all $k \leq n$, let $A_k = \{\omega : T(\omega) = k\}$. As T is a random variable, A_k is an event, and by the definition of stopping times, we see that if $\omega \in A_k$, then $[\omega]_k \in A_k$. This proves that $A_k \in \mathcal{A}_k$. As $\mathcal{A}_k \subseteq \mathcal{A}_n$ and

$$\{\omega : T(\omega) \leq n\} = \bigcup_{k \leq n} A_k \, ,$$

we see that $\{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$.

For the converse, assume that $A_n = \{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$ for all n. Since

$$\{\omega : T(\omega) = n\} = A_n \setminus A_{n-1}$$

we see that $\{\omega : T(\omega) = n\} \in \mathcal{A}_n$. This means that if $T(\omega) = n$, then $T(\omega') = n$ for all $\omega' \in [\omega]_n$, and hence T is a stopping time. □

The next thing we want to look at, is the σ-algebra $\mathcal{A}_T$ generated by a stopping time T. The intuitive idea is that it consists of sets that are observable at the time T in the following sense.

Definition 7 *Assume that T is an $\{\mathcal{A}_n\}$-stopping time. An event A is* observable at T *if whenever $T(\omega) = n$ and $\omega \in A$, then $[\omega]_n \subseteq A$. We let $\mathcal{A}_T$ denote the collection of all sets observable at T.*

Just as for non-random times, we have:

Proposition 8 *$\mathcal{A}_T$ is a σ-algebra.*

Proof: The proof is a slightly more complicated version of the proof of Proposition 3. We again check the three conditions for a σ-algebra.

(i) As $\emptyset$ doesn't have any element, it clearly satisfies the criterion for being in $\mathcal{A}_T$.

(ii) Assume $A \in \mathcal{A}_T$; we must prove that $A^c \in \mathcal{A}_T$. Let $\omega \in A^c$ and put $n = T(\omega)$. We must show that $[\omega]_n \subseteq A^c$. Assume not; then there is an $\omega' \in [\omega]_n$ such that $\omega' \in A$. Since T is a stopping time and $\omega' \sim_n \omega$, we have $T(\omega') = T(\omega) = n$. But since $\omega' \in A \in \mathcal{A}_T$, this means that $[\omega']_n \subseteq A$, which is impossible as $\omega \in [\omega']_n$ and $\omega \in A^c$.

(iii) Assume that $A_k \in \mathcal{A}_T$ for all $k \in \mathbb{N}$; we must show that $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}_T$: Let $\omega \in \bigcup_{k\in\mathbb{N}} A_k$ and put $n = T(\omega)$. Clearly, $\omega \in A_k$ for some $k \in \mathbb{N}$, and since $A_k \in \mathcal{A}_T$, this means that $[\omega]_n \subseteq A_k$. But then $[\omega]_n \subseteq \bigcup_{k\in\mathbb{N}} A_k$, which is what we had to prove. □

We now want to find a description of $\mathcal{A}_T$ that is easier to generalize to other filtrations.

Proposition 9 *Assume that T is an $\{\mathcal{A}_n\}$-stopping time. Then $A \in \mathcal{A}_T$ if and only if $A \cap \{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$ for all n.*

Proof: Assume that $A \in \mathcal{A}_T$. Given an $\omega \in A \cap \{\omega : T(\omega) \leq n\}$, we must show that $[\omega]_n \subseteq A \cap \{\omega : T(\omega) \leq n\}$. Assume $\omega' \in [\omega]_n$. We clearly have $T(\omega) = k$ for some $k \leq n$, and since $\omega' \in [\omega]_n \subseteq [\omega]_k$, we see that $T(\omega') = k$. Hence $\omega' \in \{\omega : T(\omega) \leq n\}$. Moreover, since $\omega' \in [\omega]_k$ and $T(\omega) = k$, the assumption that $\omega \in A$, implies that $\omega' \in A$ (here we are using that $A \in \mathcal{A}_T$). But then $\omega' \in A \cap \{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$, as we had to prove.

Assume for the converse that $A \cap \{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$ for all n. We must prove that $A \in \mathcal{A}_T$; i.e. we have to show that if $\omega \in A$ and $T(\omega) = n$, then $[\omega]_n \subseteq A$. But since $\omega \in A \cap \{\omega : T(\omega) \leq n\} \in \mathcal{A}_n$, we have $[\omega]_n \subseteq A \cap \{\omega : T(\omega) \leq n\}$, and in particular $[\omega]_n \subseteq A$. □

Concluding remarks: What have we done in this note? We have introduced a natural filtration $\{\mathcal{A}_n\}$ and defined stopping times and σ-algebras generated by stopping times in ways that make intuitive sense (definitions 2 and 7). Unfortunately, these intuitive definitions do not make sense for more general filtrations, but we have proved that they are equivalent to descriptions (propositions 6 and 9) which can easily be extended to the general case. Hence for general filtrations we define stopping times and the σ-algebras they generate by the descriptions in these propositions (just as the textbook does):

Definition 10 *Assume $\{\mathcal{F}_n\}$ is a filtration (i.e. an increasing sequence of σ-algebras such that $\mathcal{F}_n \subseteq \mathcal{F}$ for all n). A random variable $T\colon \Omega \to \mathbb{N} \cup \{\infty\}$ is a* stopping time with respect to *$\{\mathcal{F}_n\}$ if and only if $\{\omega : T(\omega) \leq n\} \in \mathcal{F}_n$ for all n.*

Similarly for the σ-algebra generated by a stopping time:

Definition 11 *Assume $\{\mathcal{F}_n\}$ is a filtration and T is a stopping time with respect to $\{\mathcal{F}_n\}$. The* σ-algebra $\mathcal{F}_T$ generated by T *consists of all event A such that $A \cap \{\omega : T(\omega) \leq n\} \in \mathcal{F}_n$ for all n.*

It is not at all obvious that the intuition we have from the $\{\mathcal{A}_n\}$-case carries over to the general case, but experience proves that it does — and it even carries over to the still more complicated case where the timeline is continuous and not discrete.