STK-MAT3710/4710: Solution to Mandatory Assignment, Fall 2022

Problem 1. a) As $X_n(0) = \sqrt{n}$ for all n, and $X(0) = 0$, the sequence does not converge at 0, and hence it does not converge pointwise.

b) For any $x > 0$, we see that $X_n(x) = 0$ for $n > \frac{1}{x}$, and hence $\lim_{n\to\infty} X_n(x) = 0$. This means that convergence only fails on the set $\{0\}$ which has probability 0, and hence $\{X_n\}$ converges to 0 a.s.

c) For any $\epsilon > 0$, we have

$$\{x \in \omega \,:\, |X_n(\omega)| \geq \epsilon\} \subseteq \left[0, \frac{1}{n}\right],$$

and thus

$$P(\{x \in \omega \,:\, |X_n(\omega)| \geq \epsilon\}) \leq P\left(\left[0, \frac{1}{n}\right]\right) = \frac{1}{n} \to 0$$

This shows that the sequence converges to 0 in probability. One may also use b) and the fact that a.s. convergence implies convergence in probability (Proposition 4.5).

d) We have

$$E[|X_n - 0|] = E[X_n] = \sqrt{n} \cdot \frac{1}{n} = \frac{1}{\sqrt{n}} \to 0$$

which shows that the sequence converges to 0 in L^1.

e) We have

$$E[|X_n - 0|^2] = E[X_n^2] = (\sqrt{n})^2 \cdot \frac{1}{n} = \frac{n}{n} = 1$$

which shows that the sequence does not converge to 0 in L^2.

f) The distribution function of the constant random variable $X = 0$ is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ 1 & \text{if } x \geq 0 \end{cases}$$

The distribution function of X_n is

$$F_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ 1 - \frac{1}{n} & \text{if } 0 \leq x < \sqrt{n} \\ \\ 1 & \text{if } x \geq \sqrt{n} \end{cases}$$

We see that $F_n(x) \to F(x)$ for all x, and hence $\{X_n\}$ converges to $X = 0$ in distribution. (To see this, note that for $x < 0$, $F_n(x) = F(x) = 0$ and hence the convergence is obvious. For $x \geq 0$, we see that when n gets big, $x < \sqrt{n}$, and hence $F_n(x) = 1 - \frac{1}{n}$. Consequently, $\lim_{n\to\infty} F_n(x) = 1 = F(x)$.)

Comment: Many solves this problem by combining c) and the assertion that convergence in probability implies convergence in distribution. The assertion is correct, but I can't remember that we have covered it.

Problem 2. By Lyapounov's second inequality (Corollary 3.23),

$$E[|X - X_n|^p]^{1/p} \leq E[|X - X_n|^q]^{1/q},$$

and hence

$$E[|X - X_n|^p] \leq E[|X - X_n|^q]^{p/q} \to 0$$

as $n \to \infty$.

Problem 3. a) Differentiating, we get

$$\phi'(u) = \frac{2u \cdot (\alpha + u)^2 - (\sigma^2 + u^2) \cdot 2 \cdot (\alpha + u)}{(\alpha + u)^4}$$

$$= \frac{2u(\alpha + u) - 2(\sigma^2 + u^2)}{(\alpha + u)^3} = \frac{2u\alpha - 2\sigma^2}{(\alpha + u)^3}$$

which is 0 for $u = \frac{\sigma^2}{\alpha}$. As $\phi'(u) < 0$ when $u < \frac{\sigma^2}{\alpha}$, and $\phi'(u) > 0$ when $u > \frac{\sigma^2}{\alpha}$, we see that $u = \frac{\sigma^2}{\alpha}$ is the minimum point. The minimum value is

$$\phi\left(\frac{\sigma^2}{\alpha}\right) = \frac{\sigma^2 + \left(\frac{\sigma^2}{\alpha}\right)^2}{(\alpha + \frac{\sigma^2}{\alpha})^2} = \frac{\frac{\sigma^2}{\alpha^2}(\alpha^2 + \sigma^2)}{\frac{1}{\alpha^2}(\alpha^2 + \sigma^2)^2} = \frac{\sigma^2}{\sigma^2 + \alpha^2}$$

b) Note that if $X(\omega) < \alpha$, then the left hand side of the inequality is zero while the right hand side is nonnegative, so the inequality holds. If, on the other hand, $X(\omega) \geq \alpha$, then $X(\omega) + u \geq \alpha + u > 0$, and hence $\frac{(X+u)^2}{(\alpha+u)^2} \geq 1 \geq \mathbf{1}_{\{X \geq \alpha\}}$.

c) For any $u > 0$, we know from b) that

$$P\{\omega : X(\omega) \geq \alpha\} = E\left[\mathbf{1}_{\{X \geq \alpha\}}\right] \leq E\left[\frac{(X + u)^2}{(\alpha + u)^2}\right] = \frac{\sigma^2 + u^2}{(\alpha + u)^2}$$

By a), the smallest possible value of the expression on the right is $\frac{\sigma^2}{\sigma^2 + \alpha^2}$, and hence

$$P\{\omega : X(\omega) \geq \alpha\} \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}$$

d) Put $X = Y - \mu$; then X has expectation 0 and variance σ^2, and hence by c)

$$P\{\omega \,:\, X(\omega) \geq \alpha\} \leq \frac{\sigma^2}{\sigma^2 + \alpha^2},$$

which is equivalent to

$$P\{\omega : Y(\omega) \geq \alpha + \mu\} \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}.$$

(This inequality is often called *Cantelli's Inequality.*)

Problem 4. a) For S_n to be 0, we need the X_k's to take the values 1 as -1 equally many times, and this is impossible if n is odd. If n is even, there are $\binom{n}{n/2}$ ways in which to choose $n/2$ positive values among n possible, and each such combination happens with probability $p^{n/2}(1-p)^{n/2}$. Hence

$$P[S_n = 0] = \begin{cases} \binom{n}{n/2} p^{n/2}(1-p)^{n/2} & \text{if } n \text{ is even} \\ \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

b) Let

$$A_n = \{\omega \,:\, S_{2n}(\omega) = 0\}$$

(note the shift from $2n$ to n). If we can show that $\sum_{n=0}^{\infty} P(A_n) < \infty$, the Borel-Cantelli Lemma tells us that the probability that S_n is 0 infinitely many times equals zero. As

$$P(A_n) = \binom{2n}{n} p^n (1-p)^n$$

by part a), we can use the Ratio Test to check if $\sum_{n=0}^{\infty} P(A_n)$ converges:

$$\lim_{n\to\infty} \frac{P(A_{n+1})}{P(A_n)} = \lim_{n\to\infty} \frac{\binom{2n+2}{n+1} p^{n+1}(1-p)^{n+1}}{\binom{2n}{n} p^n (1-p)^n}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} p(1-p) = 4p(1-p)$$

A little calculus shows that $f(p) = p(1-p)$ has its maximal value $\frac{1}{4}$ at $p = \frac{1}{2}$, and since by assumption $p \neq \frac{1}{2}$, we have $p(1-p) < \frac{1}{4}$. Thus

$$\lim_{n\to\infty} \frac{P(A_{n+1})}{P(A_n)} < 1$$

which means that the series $\sum_{n=0}^{\infty} P(A_n)$ converges, and hence the probability that S_n is 0 infinitely many times is zero. (If $p = \frac{1}{2}$, one can show that with probability 1, $S_n = 0$ infinitely many times.)

Comment: It is also possible to use Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ to solve this problem, but one needs to be a little careful with the formulations as $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ means that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

and not that

$$\lim_{n\to\infty} \left(n! - \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) = 0$$

(this last limit is in fact infinite). A long as one sticks to ratio or comparison tests, it is not hard to get this to work.