28th September, 2022

STK-MAT3710/4710

Mandatory assignment 1 of 1

Submission deadline

Thursday 13th October 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

If you are taking STK-MAT3710, you can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). If you are taking STK-MAT4710, you must write your solution in LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have one attempt at the assignment, and you need to have the assignments approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

From the fall semester of 2021, there is a new regime for mandatory assignments. In the new regime, you only have one attempt at each assignment and not two as in earlier years. As the purpose of the new regime is to handle the assignments in a more efficient and pedagogical way and not to fail more students, we are putting more emphasis on effort *in the grading: As long as you have documented that you have made a serious attempt at the majority of the problems, we will pass you. The best way to document that you have tried, is, of course, to solve the problems, but you can also do it by telling us what you have tried and why it failed. We encourage you to discuss, collaborate, and help each other. Do not hesitate to contact the lecturer (preferably well in advance of the deadline) if you have problems.*

If there is a problem you cannot solve, you can still use the result freely in any subsequent problem.

Problem 1. In this problem, $\Omega = [0,1]$, $\mathcal{B}$ is the Borel σ-algebra on Ω, and P is the Lebesgue measure on $(\Omega, \mathcal{B})$. A sequence $\{X_n\}$ of random variables on $(\Omega, \mathcal{B}, P)$ is defined by

$$X_n(x) = \begin{cases} \sqrt{n} & \text{if } x \in [0, \frac{1}{n}] \\ \\ 0 & \text{otherwise} \end{cases}$$

Does the sequence $\{X_n\}$ converge to $X = 0$

a) pointwise?

b) almost surely?

c) in probability?

d) in L^1?

e) in L^2?

f) in distribution?

Remember to give reason for your answers.

Problem 2. Assume $p, q \in \mathbb{R}$, $1 \leq p < q$. Show that if a sequence $\{X_n\}$ of random variables converges to X in L^q, then it converges to X in L^p.

Problem 3. a) Let $\phi : [0, \infty) \to \mathbb{R}$ be defined by

$$\phi(u) = \frac{\sigma^2 + u^2}{(\alpha + u)^2}$$

where $\alpha, \sigma > 0$. Show that the smallest value for ϕ is $\frac{\sigma^2}{\sigma^2+\alpha^2}$ obtained at $u = \frac{\sigma^2}{\alpha}$.

b) Let X be a random variable with mean $E[X] = 0$ and variance $E[X^2] = \sigma^2$. Show that if $\alpha > 0$, then for any $u \geq 0$,

$$\mathbf{1}_{\{X \geq \alpha\}} \leq \frac{(X + u)^2}{(\alpha + u)^2}$$

and use this to show that

$$P\{\omega : X(\omega) \geq \alpha\} \leq \phi(u).$$

c) Show that

$$P\{\omega : X(\omega) \geq \alpha\} \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}.$$

d) Show that if Y is a random variable with mean $E[Y] = \mu$ and variance $E[(Y - \mu)^2] = \sigma^2$, then

$$P\{\omega : Y(\omega) \geq \alpha + \mu\} \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}.$$

Problem 4. Let $\{X_k\}$ be a sequence of independent, identically distributed random variables taking the value 1 with probability p and the value -1 with probability $1 - p$. Let $S_n = \sum_{k=1}^{n} X_k$.

a) Show that

$$P[S_n = 0] = \begin{cases} \binom{n}{n/2} p^{n/2}(1-p)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

b) Show that if $p \neq \frac{1}{2}$, then for almost all ω, $S_n(\omega)$ is only 0 for finitely many n.

Good Luck!