Løsningsforslag til eksamen i STK1100, 7. juni 2007

Oppgave 1.

a) $F_X(x) = \int_0^x \frac{1}{\theta} dy = \begin{cases} 0 & x < 0 \\ x/\theta & 0 \le x \le \theta \\ 1 & \theta < x \end{cases}$

b) $EX = \int_0^\theta x f_X(x) dx = \int_0^\theta x \frac{1}{\theta} dx = \frac{1}{\theta}\left[x^2/2\right]_0^\theta = \theta/2$

$EX^2 = \int_0^\theta x^2 \frac{1}{\theta} dx = \frac{1}{\theta}\left[\frac{1}{3}x^3\right]_0^\theta = \theta^2/3$

$\text{Var}\, X = EX^2 - (EX)^2 = \theta^2/3 - \theta^2/4 = \theta^2/12$

c) $E\bar{X} = \frac{1}{n}\sum_{i=1}^n EX_i = \frac{1}{n}\sum_{i=1}^n \theta/2 = \theta/2$

$\text{Var}\, \bar{X} = \frac{1}{n^2}\sum_{i=1}^n \text{Var}\, X_i = \frac{1}{n^2} \cdot n\ \theta^2/12 = \theta^2/12n$

d) Fra c) har vi at

$E(2\bar{X}) = 2E(\bar{X}) = 2 \cdot \theta/2 = \theta$

Bruker derfor $2\bar{x}$ som estimat for θ.

e) $P(\bar{X} \le 0,75) = P\big(\frac{\bar{X}-\theta/2}{\theta/\sqrt{12\cdot 48}} \le \frac{0,75-\theta/2}{\theta/24}\big) \approx \varphi\big(\frac{18}{\theta} - 12\big)$

f) Et anslag på θ ifølge d) er $2\bar{x} = 2\cdot 0,6 = 1,2$. Et anslag på sannsynligheten i e) er da

$\varphi\big(\frac{18}{1.2} - 12\big) = \varphi(15-12) = \varphi(3) = 0,9987$

$\bar{X} \le 0,75$ betyr at i gjennomsnitt er beløpet på regningen minst 1/3 høyere enn det korrekte beløp. Sannsynligheten for dette er svært nær 1.

Oppgave 2

a) $k\int_0^1\int_0^x (x-y^2)dy\,dx = k\int_0^1 [xy - y^3/3]_0^x dx$

$= k\int_0^1 (x^2 - x^3/3)dx = k[\frac{x^3}{3} - \frac{x^4}{12}]_0^1 = k[\frac{1}{3} - \frac{1}{12}] = k \cdot \frac{3}{12} = k/4$

Dermed er $k = 4$

b) $f_X(x) = 4\int_0^x (x-y^2)dy = 4[xy - y^3/3]_0^x = 4(x^2 - x^3/3) = \frac{4}{3}x^2(3-x)$,
$0 \le x \le 1$

$f_Y(y) = 4\int_y^1 (x-y^2)dx = 4[x^2/2 - y^2x]_y^1 = 4(1/2 - y^2 - y^2/2 + y^3)$
$= 2(1 - 3y^2 + 2y^3), \quad 0 \le y \le 1$

Siden vi opplagt ikke har at $f(x,y) = f_X(x) \cdot f_Y(y)$ for alle $0 \le x \le 1$, $0 \le y \le 1$, er X og Y avhengige.

c) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{4(x-y^2)}{\frac{4}{3}x^2(3-x)} = \frac{3(x-y^2)}{x^2(3-x)} \qquad 0 \le y \le x$

$E(Y|X=x) = \int_0^x y\frac{3(x-y^2)}{x^2(3-x)}dy = \frac{3}{x^2(3-x)}[xy^2/2 - y^4/4]_0^x$
$= \frac{3}{x^2(3-x)}(\frac{x^3}{2} - \frac{x^4}{4}) = \frac{3x(2-x)}{4(3-x)}$

d) $EX = \int_0^1 \frac{4}{3}(3x^3 - x^4)dx = \frac{4}{3}[\frac{3}{4}x^4 - \frac{1}{5}x^5]_0^1 = \frac{4}{3}(\frac{3}{4} - \frac{1}{5}) = \frac{4}{3}(\frac{15-4}{20}) = \frac{11}{15}$

$EX^2 = \int_0^1 \frac{4}{3}(3x^4 - x^5)dx = \frac{4}{3}[3\frac{x^5}{5} - \frac{x^6}{6}]_0^1 = \frac{4}{3}(\frac{3}{5} - \frac{1}{6}) = \frac{4}{3}(\frac{18-5}{30}) = \frac{26}{45}$

$\operatorname{Var} X = \frac{26}{45} - (\frac{11}{15})^2 = \frac{26}{45} - \frac{121}{225} = \frac{130-121}{225} = \frac{9}{225} = \frac{1}{25}$

$\operatorname{sd}(X) = \frac{1}{5}$

$EY = \int_0^1 2(y - 3y^3 + 2y^4)dy = 2[y^2/2 - 3y^4/4 + 2y^5/5]_0^1$
$= 2(1/2 - 3/4 + 2/5) = \frac{2}{20}(10 - 15 + 8) = \frac{3}{10}$

$EY^2 = \int_0^1 2(y^2 - 3y^4 + 2y^5)dy = 2[y^3/3 - 3y^5/5 + 2y^6/6]_0^1$
$= \frac{2}{15}[5 - 9 + 5] = \frac{2}{15}$

$\operatorname{Var} Y = \frac{2}{15} - (\frac{3}{10})^2 = \frac{1}{300}(40 - 27) = 13/300$

$\operatorname{sd}(Y) = \sqrt{13/3}/10$

$E(XY) = 4\int_0^1\int_0^x (x^2y - xy^3)dy\,dx = 4\int_0^1 [x^2y^2/2 - xy^4/4]_0^x dx$
$= 4\int_0^1 (x^4/2 - x^5/4)dx = 4[\frac{1}{2}x^5/5 - \frac{1}{4}x^6/6]_0^1 = 4[\frac{1}{10} - \frac{1}{24}]$
$= \frac{2}{5} - \frac{1}{6} = \frac{1}{30}(12 - 5) = \frac{7}{30}$

$\rho(X,Y) = \frac{E(XY) - EX \cdot EY}{\operatorname{sd}(X) \cdot \operatorname{sd}(Y)} = \frac{7/30 - \frac{11}{15}\cdot\frac{3}{10}}{\frac{1}{5}\cdot\sqrt{\frac{13}{3}}/10} = \frac{35-33}{3\sqrt{13/3}} = \frac{2}{\sqrt{39}} = 0,32$