STK1100 Spring 2022.

Exercise 5.103 - Solution.

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$f_{X_1,X_2}(x_1,x_2) = 2(x_1+x_2)$, $0 \le x_1 \le x_2 \le 1$

Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$.

We aim to find (a) $f_{Y_1}(y_1)$ and (b) $f_{Y_2}(y_2)$.

Invert: $X_1 = \dfrac{Y_1 - Y_2}{2}$, $X_2 = \dfrac{Y_1 + Y_2}{2}$.

Jacobian: $J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

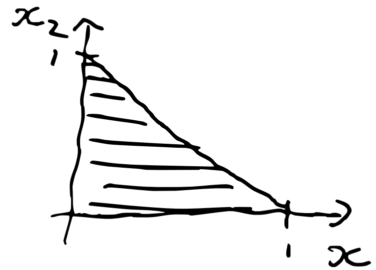
So

$$f_{Y_1,Y_2}(y_1,y_2) = J \times f_{X_1,X_2}(x_1(y_1,y_2), x_2(y_1,y_2))$$

$$= \frac{1}{2} \times 2\left(\frac{y_1-y_2}{2} + \frac{y_1+y_2}{2}\right) = y_1.$$

Domain of (y_1, y_2):

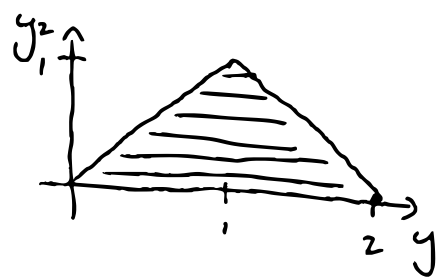
We have that $0 \le x_1 \le x_2 \le 1$.



i.e. $0 \le \frac{y_1 - y_2}{2} \le \frac{y_1 + y_2}{2} \le 1$

So $y_2 \le y_1$, $y_2 \ge 0$

and $y_1 \le 2 - y_2$



Hence

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} y_1 & \text{if } 0 \le y_2 \le y_1 \le 2 - y_2 \\ 0 & \text{otherwise.} \end{cases}$$

Marginalise:

(a)

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1, Y_2}(y_1, y_2)\, dy_2$$

$$= \int_{y_2=0}^{\min(y_1, 2-y_1)} y_1 \, dy_2 = y_1 [y_2]_{y_2=0}^{\min(y_1, 2-y_1)}$$

$$= y_1 \min(y_1, 2-y_1) = \begin{cases} y_1^2 & \text{if } \quad 0 \le y_1 < 1 \\ y_1(2-y_1) & \text{if } \quad 1 \le y_1 \le 2 \end{cases}$$

(b) $f_{Y_2}(y_2) = \int_{\mathbb{R}} f_{Y_1, Y_2}(y_1, y_2) \, dy_1$

$$= \int_{y_1=y_2}^{2-y_2} y_1 \, dy_1 = \frac{1}{2} [y_1^2]_{y_1=y_2}^{2-y_2}$$

$$= \frac{1}{2} \left[(2-y_2)^2 - y_2^2 \right] = \frac{1}{2}(4 - 4y_2)$$

$$= 2(1-y_2),$$

$$0 \le y_2 \le 1.$$
