Gammafordelingen – Momentgenererende funksjon

$X \sim \text{Gamma}(\alpha, \beta)$, dvs. tetthet

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{ellers} \end{cases}$$

$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} \, du$ er gammafunksjonen

~~(fra forrige gang!)~~

Fra definisjonen av $M_X(t)$:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \, dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(1-t\beta)x/\beta} \, dx$$

$1 - t\beta > 0$

$t < \frac{1}{\beta}$

substitusjon

$u = (1-t\beta)\frac{x}{\beta}$

$du = \frac{(1-t\beta)}{\beta} dx$

$x = \frac{\beta}{1-t\beta} u$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left[\frac{\beta u}{1-t\beta}\right]^{\alpha-1} e^{-u} \frac{\beta}{1-t\beta} \, du$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\beta^\alpha}{(1-t\beta)^\alpha} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u} \, du}_{\Gamma(\alpha)} = \underline{\underline{\frac{1}{(1-t\beta)^\alpha}}}$$

Forventning og varians:

$$M_X'(0) = E(X) \qquad M_X''(0) = E(X^2)$$

$$M_X(t) = (1 - t\beta)^{-\alpha}$$

$$M_X'(t) = +\alpha\beta(1-t\beta)^{-\alpha-1}$$

$$\underline{\underline{M_X'(0) = \alpha\beta = E(X)}}$$

$$M_X''(t) = +\alpha\beta^2(\alpha+1)(1-t\beta)^{-\alpha-2}$$

$$M_X''(0) = \alpha\beta^2(\alpha+1)$$

$$\underline{\underline{V(X)}} = \overbrace{\alpha\beta^2(\alpha+1)}^{E(X^2)} \overbrace{-\alpha^2\beta^2}^{-\{E(X)\}^2}$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \underline{\underline{\alpha\beta^2}}$$

evt. via

$R_X(t) = \ln M_X(t)$

som på slides!