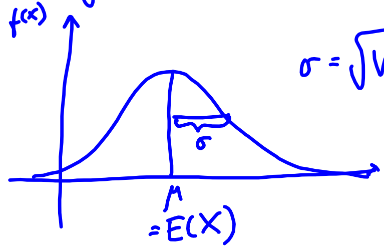
Recap kap. 4. 3 Normalfordelingen

$$X \sim N(\mu, \sigma^2)$$



$\sigma = \sqrt{V(X)}$

$$\frac{X-\mu}{\sigma} = Z \sim N(0,1)$$

standard normal

(tabell)

Momentgenererende funksjon for normalford.

$X \sim N(\mu, \sigma^2)$ $\quad Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

Da har vi også $\boxed{X = \sigma Z + \mu}$ ⊛

Finner den mgf til Z:

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2tz)} \, dz$$

triks!

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2tz + t^2) + t^2/2} \, dz$$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}}_{N(t,1)\text{ - tettheten}} \, dz}_{=1} = e^{t^2/2}$$

Ser så på X:

$$M_X(t) = E(e^{tX}) \overset{⊛}{=} E\left(e^{t(\sigma Z + \mu)}\right)$$

$$= e^{t\mu} E\left(e^{(t\sigma)Z}\right) = e^{t\mu} M_Z(\sigma t)$$

$$= e^{t\mu} \cdot e^{(\sigma t)^2/2} = e^{t\mu + \sigma^2 t^2/2}$$

4.6 Normalfordelingsplott

Empirisk fordeling og empiriske persentiler

Har en stok. var. X

Kum. ford. $F(x) = P(X \leq x)$ ukjent

Antar at vi observerer verdier av X n ganger

Får obs. $x_1 \; x_2 \; \dots \; x_n$

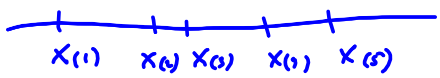
Vil "gjette på" hva F er (estimere)

Empirisk : Bruker

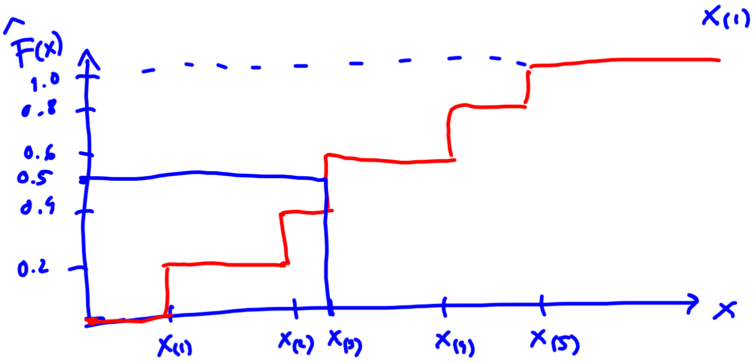
$$\hat{F}(x) = \frac{\text{antall } x_i\text{-er} \leq x}{n} = \frac{\#(x_i \leq x)}{n}$$

Dette kalles empirisk kumulativ fordeling

Illustrasjon med $n = 5$



observasjonene ordnet i stigende rekkefølge



hvert hopp $= \frac{1}{n}$

Generelt:
empiriske persentiler er gitt som følger (boken) :

$\left(\frac{i - 1/2}{n} \right) \cdot 100\%$ - persentilen svarer til observasjon $x_{(i)}$

Eks. $n = 5$

$i = 3$ $\quad x_{(3)}$ vil være $\left(\frac{3 - 1/2}{5} \right) 100\%$ - persentil

dvs. 50 % persentil (median)

$i = 2$ $\quad x_{(2)}$ vil være $\left(\frac{2 - 1/2}{5} \right) 100\%$ - persentil

dvs. 30 % persentil

4.4 <u>Gamma - fordeling</u> + +

X stok. var. tetthet på formen

$$f(x) = \begin{cases} K \cdot x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{ellers} \end{cases}$$

Hva må K være? Vi må ha $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 0 + \int_0^{\infty} K x^{\alpha-1} e^{-x/\beta}\,dx = K \int_0^{\infty} x^{\alpha-1} e^{-x/\beta}\,dx$$

subst. $u = \frac{x}{\beta}$, $x = \beta u$, $dx = \beta\,du$

$$= K \int_0^{\infty} (\beta u)^{\alpha-1} e^{-u} \beta\,du = K \beta^{\alpha} \underbrace{\int_0^{\infty} u^{\alpha-1} e^{-u}\,du}_{\Gamma(\alpha)}$$

$$= K \cdot \beta^{\alpha} \Gamma(\alpha)$$

Må derfor ha $K = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}$

Derfor blir tettheten

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{ellers} \end{cases}$$

Egenskaper gammafunksjonen

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du \qquad \alpha > 0$$

① $\boxed{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)}$ $\qquad \alpha > 1$

Bevis:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du \overset{\text{delvis integrasjon}}{=} \left[u^{\alpha-1}(-e^{-u})\right]_0^\infty - \int_0^\infty (\alpha-1)u^{\alpha-2}(-e^{-u})\, du$$

$$= 0 + (\alpha-1)\underbrace{\int_0^\infty u^{\alpha-2} e^{-u}\, du}_{\Gamma(\alpha-1)} = (\alpha-1)\Gamma(\alpha-1)$$

② n positivt heltall $\boxed{\Gamma(n) = (n-1)!}$

"Bevis": $n = 1$

$$\Gamma(1) = \int_0^\infty u^{1-1} e^{-u}\, du = \int_0^\infty e^{-u}\, du = -e^{-u}\Big]_0^\infty = 1 = 0! \leftarrow (1-1)!$$

$n = 2$

$$\Gamma(2) \overset{①}{=} (2-1)\Gamma(2-1) = 1 \cdot \Gamma(1) = 1 = 1! \leftarrow (2-1)!$$

$n = 3$

$$\Gamma(3) \overset{①}{=} (3-1)\Gamma(3-1) = 2 \cdot \Gamma(2) = 2 = (3-1)!$$

osv.

③ $\boxed{\Gamma(1/2) = \sqrt{\pi}}$

$$\Gamma(1/2) = \int_0^\infty u^{1/2-1} e^{-u}\, du = \int_0^\infty u^{-1/2} e^{-u}\, du$$

substituer $u = \frac{x^2}{2}$, $du = x\,dx$

$$= \int_0^\infty \left(\frac{x^2}{2}\right)^{-1/2} e^{-\frac{x^2}{2}}\, x\, dx$$

$$= \sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}}\, dx = \sqrt{2}\,\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2}}\sqrt{2\pi}\underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx}_{=1} \quad \text{dette er } N(0,1)$$

$$= \sqrt{\pi}$$

Hvis jeg trenger $\Gamma(3/2)$:

$$\Gamma(3/2) = \left(\tfrac{3}{2}-1\right)\Gamma\left(\tfrac{3}{2}-1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$