Rep. 5.1

Simultanfordelinger

- hvordan to eller flere stok. variabler oppfører seg sammen

eks. karakterer i norsk og matte på vgs (diskret)

eks. høyde og armspenn (kont.)

Diskret: simultan punktsannsynlighet

$$p(x,y) = P(X = x \text{ og } Y = y)$$

samtidig!

$p(x,y) \geq 0$

$\sum_x \sum_y p(x,y) = 1$

marginal punktsanns.

$$p_X(x) = P(X = x) = \sum_y p(x,y)$$

Kont.: simultan sanns.tetthet

$$f(x,y) = \dots$$

$f(x,y) \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

marginal tetthet

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Uavhengige stok. variable X og Y:

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad \forall x,y$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

Eks. La X og Y være kont. med simultantetthet $f(x,y)$,
og la $h(x,y) = x$

$$E(h(X,Y)) = \underline{E(X)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dy}_{f_X(x)}\, dx$$

$$= \underline{\int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

Eks. 5.9. (forts.)

X og Y ~ $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{ellers} \end{cases}$

Vi så at

$f_X(x) = \begin{cases} \frac{6}{5}x + \frac{2}{5} & 0 \le x \le 1 \\ 0 & \text{ellers} \end{cases}$ $\quad f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5} & 0 \le y \le 1 \\ 0 & \text{ellers} \end{cases}$

Vi finner

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\left(\frac{6}{5}x + \frac{2}{5}\right)dx = \int_0^1 \left(\frac{6}{5}x^2 + \frac{2}{5}x\right)dx$$

$$= \frac{2}{5}x^3 + \frac{1}{5}x^2 \Big]_0^1 = \frac{2}{5} + \frac{1}{5} = \underline{\underline{\frac{3}{5}}}$$

Tilsvarende

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \cdots = \underline{\underline{\frac{3}{5}}}$$

Vi får f.eks.

$$E(X + 2Y + 1) = E(X) + 2E(Y) + 1 = \frac{3}{5} + 2 \cdot \frac{3}{5} + 1 = \underline{\underline{\frac{14}{5}}}$$

Forventningen til et produkt ved uavhengighet

La X og Y være kont. uavh. variabler $\sim f(x,y)$

Da har vi

$$f(x,y) = f_X(x) \cdot f_Y(y) \qquad \forall\, x,y$$

Vi får

$$E(\underbrace{g(X)\cdot h(Y)}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f(x,y)\, dx\, dy$$

$$\overset{\text{uavh.}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f_X(x) f_Y(y)\, dx\, dy$$

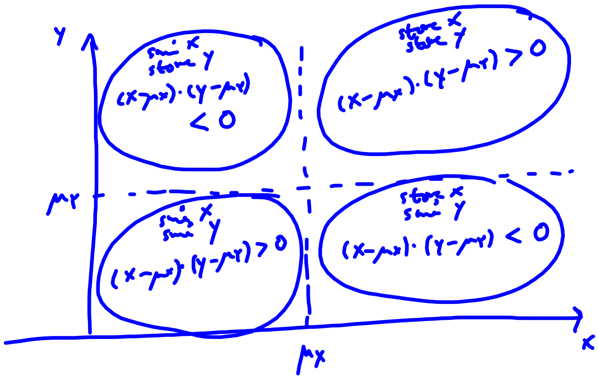
$$= \int_{-\infty}^{\infty} h(y) f_Y(y) \underbrace{\int_{-\infty}^{\infty} g(x) f_X(x)\, dx}_{E(g(X))}\, dy$$

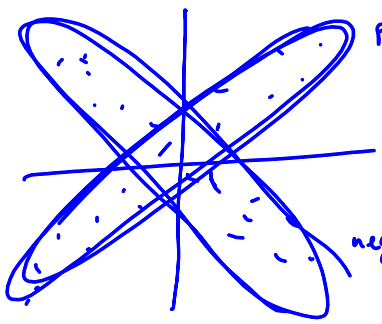
$$= E(g(X)) \cdot \underbrace{\int_{-\infty}^{\infty} h(y) f_Y(y)\, dy}_{E(h(Y))}$$

$$= E(g(X)) \cdot E(h(Y))$$

Kovarians

$$\mathrm{Cov}(X,Y) = E\big((X-\mu_X)\cdot(Y-\mu_Y)\big)$$





positiv samvariasjon
(Cov > 0)

negativ samvariasjon
(Cov < 0)

Eks. 5.4 (forts.)

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{ellers} \end{cases}$$

$E(X) = \frac{3}{5} = E(Y)$

Vi har $Cov(X,Y) = E(X \cdot Y) - E(X)E(Y)$

Vi får

$$E(XY) = \int_0^1\int_0^1 xy\frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^1\int_0^1 \left(\frac{6}{5}x^2y + \frac{6}{5}xy^3\right)dx\,dy$$

$$= \int_0^1 \left[\frac{2}{5}x^3y + \frac{3}{5}x^2y^3\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{2}{5}y + \frac{3}{5}y^3\right)dy$$

$$= \left[\frac{1}{5}y^2 + \frac{3}{20}y^4\right]_0^1$$

$$= \frac{1}{5} + \frac{3}{20} = \underline{\underline{\frac{7}{20}}}$$

Det gir $Cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

$$= \frac{7}{20} - \frac{3}{5} \cdot \frac{3}{5} = \underline{\underline{\frac{-1}{100}}}$$

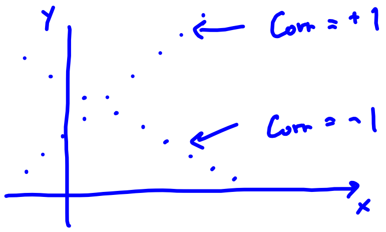
Korrelasjonskoeffisient

$$\text{Corr}(aX+b,\, cY+d) = \frac{\text{Cov}(aX+b,\, cY+d)}{\sqrt{V(aX+b)}\sqrt{V(cY+d)}}$$

$$= \frac{ac\,\text{Cov}(X,Y)}{\sqrt{a^2 V(X)}\sqrt{c^2 V(Y)}}$$

$$= \frac{ac\,\text{Cov}(X,Y)}{|a|\sigma_X \cdot |c|\sigma_Y} = \begin{cases} \text{Corr}(X,Y) & a\cdot c > 0 \\ -\text{Corr}(X,Y) & a\cdot c < 0 \end{cases}$$

Mål for lineær sammenheng:



$$\text{Corr}(X,\, aX+b) = \frac{\text{Cov}(X,\, aX+b)}{\sqrt{V(X)}\sqrt{V(aX+b)}}$$

$$= \frac{a\,\text{Cov}(X,X)}{\sqrt{V(X)}\,|a|\sqrt{V(X)}}$$

$$= \frac{a}{|a|}\frac{V(X)}{V(X)} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$$

Uavh. $\Rightarrow$ Corr = 0

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X\,\sigma_Y}$$

$$\text{Cov}(X,Y) = \iint (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$$

$$\overset{\text{uavh.}}{=} \iint (x-\mu_X)(y-\mu_Y) f_X(x) f_Y(y)\,dx\,dy$$

$$= \int (x-\mu_X) f_X(x) \underbrace{\int (y-\mu_Y) f_Y(y)\,dy}\, dx$$

$= 0$ når uavhengige