STK2100

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Exer 2.11

$$
\begin{aligned}
Q_{n+1}(\hat{\boldsymbol{\beta}}_{(n+1)}) =&(\boldsymbol{y}_{n+1}-\boldsymbol{X}_{n+1}\hat{\boldsymbol{\beta}}_{(n+1)})^T(\boldsymbol{y}_{n+1}-\boldsymbol{X}_{n+1}\hat{\boldsymbol{\beta}}_{(n+1)})\\
=&(\boldsymbol{y}_{n+1}-\boldsymbol{X}_{n+1}[\hat{\boldsymbol{\beta}}_{(n)}+\boldsymbol{k}_n e_{n+1}])^T(\boldsymbol{y}_{n+1}-\boldsymbol{X}_{n+1}[\hat{\boldsymbol{\beta}}_{(n)}+\boldsymbol{k}_n e_{n+1}])\\
=&(\boldsymbol{y}_{n}-\boldsymbol{X}_{n}[\hat{\boldsymbol{\beta}}_{(n)}+\boldsymbol{k}_n e_{n+1}])^T(\boldsymbol{y}_{n}-\boldsymbol{X}_{n}[\hat{\boldsymbol{\beta}}_{(n)}+\boldsymbol{k}_n e_{n+1}])+\\
&(y_{n+1}-\boldsymbol{x}_{n+1}^T[\hat{\boldsymbol{\beta}}_{(n)}+\boldsymbol{k}_n e_{n+1}])^2\\
=&(\boldsymbol{y}_n-\boldsymbol{X}_n\hat{\boldsymbol{\beta}}_{(n)})^T(\boldsymbol{y}_n-\boldsymbol{X}_n\hat{\boldsymbol{\beta}}_{(n)})+\boldsymbol{k}_n^T\boldsymbol{X}_n^T\boldsymbol{X}_n\boldsymbol{k}_n e_{n+1}^2-\\
&2(\boldsymbol{y}_n-\boldsymbol{X}_n\hat{\boldsymbol{\beta}}_{(n)})^T\boldsymbol{X}_n\boldsymbol{k}_n e_{n+1}+\\
&(y_{n+1}-\boldsymbol{x}_{n+1}^T\hat{\boldsymbol{\beta}}_{(n)})^2-\\
&2(y_{n+1}-\boldsymbol{x}_{n+1}\hat{\boldsymbol{\beta}}_{(n)})\boldsymbol{x}_{n+1}^T\boldsymbol{k}_n e_{n+1}+(\boldsymbol{k}_n^T\boldsymbol{x}_{n+1})^2e_{n+1}^2\\
=&Q_n(\hat{\boldsymbol{\beta}}_n)+[\boldsymbol{k}_n^T\boldsymbol{X}_n^T\boldsymbol{X}_n\boldsymbol{k}_n+1-2\boldsymbol{x}_{n+1}^T\boldsymbol{k}_n+(\boldsymbol{k}_n^T\boldsymbol{x}_{n+1})^2]e_{n+1}^2\\
=&Q_n(\hat{\boldsymbol{\beta}}_n)+[h^2\boldsymbol{x}_{n+1}^T\boldsymbol{V}_n\boldsymbol{x}_{n+1}+1-2h\boldsymbol{x}_{n+1}^T\boldsymbol{V}_n\boldsymbol{x}_{n+1}+h^2(\boldsymbol{x}_{n+1}^T V\boldsymbol{x}_{n+1})^2]e_{n+1}^2\\
=&Q_n(\hat{\boldsymbol{\beta}}_n)+[h^2\frac{1-h}{h}+1-2h\frac{1-h}{h}+h^2\frac{(1-h)^2}{h^2}]e_{n+1}^2\\
=&Q_n(\hat{\boldsymbol{\beta}}_n)+he_{n+1}^2
\end{aligned}
$$

where we have used that $(\boldsymbol{y}_n-\boldsymbol{X}_n\hat{\boldsymbol{\beta}}_{(n)})^T\boldsymbol{X}_n=0$ due to orthogonality and that we can rewrite the definition of h to

$$
\boldsymbol{x}_{n+1}^T\boldsymbol{V}_n\boldsymbol{x}_{n+1}=\frac{1-h}{h}
$$