UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	STK 1130 — Modelling by stochastic processes.
Day of examination:	Friday, June 11, 2004.
Examination hours:	14.30 – 17.30.
This examination set consists of 3 pages.	
Appendices:	None.
Permitted aids:	Calculator, List of formulas for ST 100, Rottmans formelsamling.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

In Per's house there is a staircase with M steps between the first flor, step no 0, and the second floor, step no M. When Per goes up the staircase he normally moves fewer than M times, since he now and then takes two steps at a time.

Let $X_n = j$ if Per after n moves has reached step no j, $n = 0, 1, 2, \ldots$.

Assume $\{X_n, n \geq 0\}$ is a time homogeneous Markov chain with state space $S = \{0, 1, 2, \ldots, M\}$ and transition probabilities

$$p_{ji} = \begin{cases} p & \text{if } j = i + 1 \\ q = 1 - p & \text{if } j = i + 2,\ i = 0, 1, 2, \ldots, M - 2, \\ 0 & \text{elsewhere} \end{cases}$$

$p_{M,M-1} = 1$ and $p_{MM} = 1$.

a) Define the terms class, period, recurrent and transient.
   Show that recurrence is a class property.

b) State the classes of the Markov chain, the periods of the classes and if the classes are recurrent or transient. State the reasons for your answers.

(Continued on page 2.)

c) What is the probability of Per to go up the staircase without treading on step no 2?

d) You shall find the probability of Per to go up the staircase without treading on step no $M-1$ (because it is squeaking). Introduce the notation u_i for the probability of Per to go up the staircase without treading on step no $M-1$ given he is standing on step no i, $i = 0, 1, 2, \ldots, M-1$.
Give a set of equations to determine $u_0, u_1, u_2, \ldots, u_{M-1}$.
Find u_{M-2}, u_{M-3} and u_{M-4} ($u_{M-1} = 0$).

e) It can be shown that the probability of Per to go up the whole staircase without treading on step no $M-1$ is

$$u_0 = \big(q + (-q)^M\big)/(1+q).$$

Calculate u_0 for $M = 10$ and $M = 11$ when (i) $q = 0.2$ and when (ii) $q = 0.9$.
Comment on the difference in the results between (i) and (ii). Have you a reasonable explanation of the difference?

Problem 2.

In a factory hall there are N machines and M ($\leq N$) workers who maintain the machines. When a machine breaks down, repair is immediately started if one of the workers is unoccupied. If all workers are occupied, the repair must wait until one is unoccupied.

In a small time period of length Δt each machine has a probability of $\lambda\Delta t + o(\Delta t)$ to break down. A machine under repair has a probability of $\mu\Delta t + o(\Delta t)$ to be back in working order in a small time period of length Δt.

Let $X(t)$ be the number of machines which are broken at time t. You can assume that $\{X(t), t \geq 0\}$ is a time continuous and time homogeneous Markov process with discrete state space $S = \{0, 1, 2, \ldots, N\}$, and let

$$p_{ji}(t) = P(X(t+s) = j | X(s) = i), \qquad i, j \in S.$$

a) Give an intuitive explanation of:

$$p_{i+1,i}(\Delta t) = (N-i)\lambda\Delta t + o(\Delta t), \quad i = 0, 1, \ldots, N-1$$

$$p_{i-1,i}(\Delta t) = \begin{cases} i\mu\Delta t + o(\Delta t), & i = 1, 2, \ldots, M \\ M\mu\Delta t + o(\Delta t), & i = M+1, \ldots, N \end{cases}$$

$$p_{ji}(\Delta t) = o(\Delta t) \quad \text{if } |j-i| > 1$$

In the rest of this problem $M = N$.

b) Define the meaning of the (infinitesimal) generator matrix, Q, for a time homogeneous Markov process.
Give Q for the process $\{X(t), t \geq 0\}$.

(Continued on page 3.)

c) The process $\{X(t), t \geq 0\}$ has a stationary distribution, $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)^T$. As well-known $\boldsymbol{\pi}$ is solution of the system of equations

$$Q\boldsymbol{\pi} = \mathbf{0}, \quad \sum_{i=0}^{N} \pi_i = 1.$$

Give the equations which determine $\boldsymbol{\pi}$.

d) The system of equations in c) can be solved by introducing the generating function $\Pi(s) = \sum_{i=0}^{N} s^i \pi_i$. From the equations in c) one obtains after some calculation that $\Pi(s)$ satisfies the differential equation

$$\Pi'(s) = \frac{N\lambda}{\lambda s + \mu} \Pi(s)$$

with side condition $\Pi(1) = 1$. (You shall *not* show this.)
Find $\Pi(s)$ by solving the differential equation.
What is then the stationary distribution $\boldsymbol{\pi}$?

Problem 3.

Here we will consider the special case of Problem 2 where $M = N = 2$. The generator matrix in this special case is

$$Q = \begin{pmatrix} -2\lambda & \mu & 0 \\ 2\lambda & -(\lambda + \mu) & 2\mu \\ 0 & \lambda & -2\mu \end{pmatrix}.$$

It can be shown that $Q = H\Lambda H^{-1}$, where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\lambda + \mu) & 0 \\ 0 & 0 & -2(\lambda + \mu) \end{pmatrix},$$

$$H = \begin{pmatrix} \mu^2 & -\mu & 1 \\ 2\lambda\mu & \mu - \lambda & -2 \\ \lambda^2 & \lambda & 1 \end{pmatrix} \quad \text{and} \quad H^{-1} = \frac{1}{(\lambda + \mu)^2} \begin{pmatrix} 1 & 1 & 1 \\ -2\lambda & \mu - \lambda & 2\mu \\ \lambda^2 & -\lambda\mu & \mu^2 \end{pmatrix}$$

Let $P(t) = (p_{ji}(t))$ be the matrix of transition probabilities. From the textbook it is known that $P(t)$ can be expressed by Q,

$$P(t) = e^{Qt} = I + \sum_{k=1}^{\infty} Q^k \cdot \frac{t^k}{k!}.$$

a) Express $P(t)$ as simple as possible by H and H^{-1}.

b) Find $\lim_{t \to \infty} P(t)$ expressed by λ and μ.

c) Write down $p_{j0}(t)$, $j = 0, 1, 2$ in terms of λ and μ. Do you recognize the probability distribution?

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