UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK2130 — Modelling by Stochastic Processes. Solution.

Day of examination: Wednesday June 11th 2014.

Examination hours: 14.30 – 18.30.

This problem set consists of 4 pages.

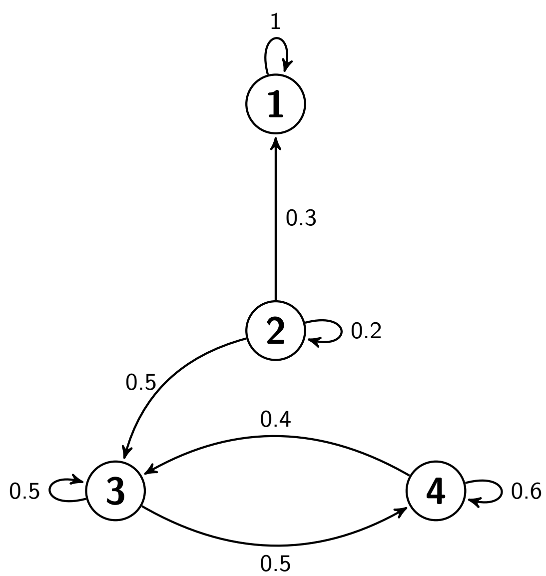
Appendices: None.

Permitted aids: Approved calculator. "Formelsamling" for STK1100 and STK1110

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a) The diagram is given by:



b) As one observes from the diagram the communicating classes are given by $\{1\}$, $\{2\}$ and $\{3, 4\}$. $\{1\}$ and $\{3, 4\}$ are closed.

c) Finite and closed classes are recurrent. Hence, $\{1\}$ and $\{3, 4\}$ are recurrent. Since $\{2\}$ is not closed, it must be transient. We see from the diagram that $P_{33}^n > 0$ for all $n = 1, 2, \ldots$. Hence, the state $i = 3$ is aperiodic.

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d) We have for all $i, j = 1, \ldots, 4$

$$P(X_2 = j | X_0 = i) = P^2_{ij}.$$

Here P^2_{ij} are the entries obtained by matrix multiplying the transition probability matrix P with itself. This leads to

$$\mathbf{PP} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0.36 & 0.04 & 0.35 & 0.25 \\ 0 & 0 & 0.45 & 0.55 \\ 0 & 0 & 0.44 & 0.56 \end{vmatrix}$$

e) When $X_0 = 3$ or $X_0 = 4$ the process starts in a recurrent and finite class with period 1. Hence, this Markov chain is ergodic and the stationary probabilities π_3 and π_4 satisfy the following equations

$$\pi_3 = \pi_3\, 0.5 + \pi_4\, 0.4$$
$$\pi_3 + \pi_4 = 1$$

This gives $\pi_3 = 4/9$ and $\pi_4 = 5/9$.

f) Starting in state 2 we shall find the expected time, ν_2, until entering one of the recurrent states. We have

$$\nu_2 = 1 + \nu_2\, 0.2,$$

which gives $\nu_2 = 1.25$. Alternatively, we have

$$\nu_2 = (1 - 0.2)^{-1} = 1/0.8 = 1.25.$$

g) Starting in state 2 we shall find the probability, q_2, of ultimate absorption in the states $\{3, 4\}$. We have

$$q_2 = q_2\, 0.2 + 0.5,$$

which gives $q_2 = 5/8$.

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Problem 2

a) See page 374 in Ross (2010). Essentially, all times to births and deaths are independent and exponentially distributed random variables. When there are n persons in the system the birth rate is $\lambda_n, n = 0, 1, \ldots$ whereas the death rate is $\mu_n, n = 1, 2, \ldots$.

b) See page 392 in Ross (2010). By letting the rate of leaving a state be equal to the rate of entering the same state under stationarity we arrive at the following balance equations

$$\lambda_0 P_0 = \mu_1 P_1$$

$$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}, \quad n = 1, 2, \ldots$$

c) The equality

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n = 0, 1, \ldots$$

follows since under stationarity the left hand side is the rate of moving one step upwards from n to $n+1$ whereas the right hand side is the rate of moving one step downwards from $n+1$ to n. This argument is not given in Ross (2010). The argument in Ross (2010) is given on page 392. Assume that the equality above is true for $n-1$. Hence,

$$\lambda_{n-1} P_{n-1} = \mu_n P_n, \quad n = 1, 2, \ldots$$

Adding this to the general equality given in b) we get the desired equality for n.

d) The last equation in c) can be written

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1}, \quad n = 1, 2, \ldots$$

By repeated use of this equation we get

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1} P_0$$

By using the fact that $\sum_{n=0}^{\infty} P_n = 1$ we get

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1}}$$

Hence,

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1[1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1}]}, \quad n = 1, 2, \ldots$$

(Continued on page 4.)

A necessary and sufficient condition for these limiting probabilities to exist is

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1} < \infty$$

e) $\lambda_n = \lambda/(n+1), n = 0, 1, \ldots$ is strictly decreasing in n. Hence, in this model customers are discouraged from joining long queues. Accordingly we have a queueing model with discouragement. Introduce $\rho = \lambda/\mu$. Then

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_2\mu_1} = \sum_{n=1}^{\infty} \rho^n/n! = \exp(\rho) - 1 < \infty$$

Hence, the limiting distribution for this model always exists irrespective of the values of λ and μ. We have

$$P_n = \frac{\rho^n}{n!(1+\exp(\rho)-1)} = \frac{\rho^n}{n!}\exp(-\rho), \quad n = 0, 1, \ldots$$

This is a Poisson distribution with parameter $\rho = \lambda/\mu$.

END