UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by Stochastic Processes

Day of examination: Friday, June 9th, 2017

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator.
"Formelsamling til STK1100 og STK1110"

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a discrete time Markov Chain with state space $\{1, 2, 3, 4, 5\}$. The matrix of one-step transition probabilities is

$$P = \begin{bmatrix} p & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & q \\ 0 & 0 & 0 & r & 1/2 \end{bmatrix}$$

a

Determine p, q and r.

b

What is the probability, starting from state 2, to be in state 2 after 2 transitions? And starting from state 3?

c

Define a recurrent and a transient state. Which are the transient and the recurrent states for this Markov Chain?

d

Compute the long-run proportions for this Markov Chain

e

What is the probability, starting from state 1, to eventually enter in $\{4{,}5\}$?

(Continued on page 2.)

Problem 2

Consider a Markov Chain defined on the vertexes of a triangular. Suppose that the process can move anti-clockwise to the next vertex with probability p and clockwise to the next vertex with probability $1-p$. Here $0 < p < 1$.

a

Write the matrix of one-step transition probabilities for this Markov Chain.

b

Show that the Markov Chain is ergodic and compute the limiting probabilities.

c

Compute for n large the probabilities $P[X_n = 1, X_{n+1} = 2]$ and $P[X_n = 2, X_{n+1} = 1]$.

d

For which p the Markov chain is reversible?

Problem 3

In a resort for downhill skiing customers arrives to a ski-lift station following a Poisson process with mean 4 per minute. A new lift arrives every 5 seconds, and if there is a person waiting it is occupied (there is only one place per lift), otherwise it will leave empty. If we denote with t_i the time at which the i-th lift is available, $t_i = 5i$. Suppose that at t=0 the lift left empty.

a

What is the distribution of inter arrival times between people at the station?

b

What is the distribution of the number of customers arriving between two successive lifts?

c

What is the probability of the first lift leaving occupied and having simultaneously exactly one person waiting in the line?

d

What is the probability that the second lift leaves empty?

(Continued on page 3.)

Problem 4

a

What characterizes a birth and death process with nonnegative parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$?

b

What is the total death rate when death occurs for each member at the same exponential rate μ? And what is the formula for the total birth rate in the case that each element gives birth at the same exponential rate λ and there is an exponential rate of increase θ of the population due to an external source?

c

In the case of the last point, show that the expected size of the population at time t, $M(t) = E[X(t)]$, is

$$M(t) = \frac{\theta}{\lambda - \mu}[e^{(\lambda-\mu)t} - 1] + ie^{(\lambda-\mu)t} \tag{1}$$

supposing that the population size at 0 is equal to i and $\lambda \neq \mu$.

(hint: a possible way is to proceed as follows:
- compute $M(t+h)$ as a conditional expectation $E[X(t+h)] = E[E[X(t+h)|X(t)]]$ (the probabilities of the three events with non-negligible probabilities $X(t+h) = X(t)+1$, $X(t+h) = X(t)-1$ and $X(t+h) = X(t)$ can be derived directly from the definition of Poisson process);
- compute the difference quotient $\frac{M(t+h)-M(t)}{h}$;
- take the limit ($h \to 0$) and solve the differential equation (remember that $M(0) = i$.)

d

Interpret equation (1), in particular the contributions of the terms $\frac{\theta}{\lambda-\mu}[e^{(\lambda-\mu)t} - 1]$ and $ie^{(\lambda-\mu)t}$ and the influence of the sign of $(\lambda - \mu)$.

THE END