SKETCH of the SOLUTIONS

Exercise 1

a p = 0; q = 3/4; r = 1/2;

b The matrix P^2 is

$$P^2 = \begin{bmatrix} 0 & 6/16 & 6/16 & 4/16 & 0 \\ 0 & 3/16 & 3/16 & 4/16 & 6/16 \\ 0 & 6/16 & 6/16 & 4/16 & 0 \\ 0 & 0 & 0 & 7/16 & 9/16 \\ 0 & 0 & 0 & 6/16 & 10/16 \end{bmatrix}$$

therefore, the desired values are those in position (2,2) and (3,2), 3/16 and 6/16, respectively.

c States 1, 2 and 3 are transient, 4 and 5 recurrent.

d $\pi_1 = 0, \pi_2 = 0, \pi_3 = 0, \pi_4 = 2/5, \pi_5 = 3/5$

e The probability is 1, because states 4 and 5 are recurrent, 1, 2 and 3 transient.

Exercise 2

a

$$P = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$$

b The chain is ergodic because all states communicate and p cannot assume values 0 and 1. By symmetry, $\pi_1 = 1/3, \pi_2 = 1/3, \pi_3 = 1/3$.

c

$$\begin{aligned} P[X_n = 1, X_{n+1} = 2] &= P[X_{n+1} = 2 | X_n = 1] P[X_n = 1] \\ &= p_{12} P[X_n = 1]. \end{aligned}$$

For n large, $P[X_n = 1] = \pi_1 = 1/3$, therefore $P[X_n = 1, X_{n+1} = 2] = p/3$. Similarly, $P[X_n = 1, X_{n+1} = 2] = (1-p)/3$.

d To be time reversible, $\pi_i p_{ij} = \pi_j p_{ji}$ should hold. Since in this case $\pi_i = \pi_j \ \forall i, j$ then it must hold $p_{ij} = p_{ji}$, i.e., $p = 1 - p$, so $p = 1/2$.

NB Note that p and $1-p$ may be inverted in the solution if the triangle was differently oriented.

Exercise 3

a The arrival times are exponential, so $f(t) = \lambda e^{-\lambda t} = 4e^{-4t}$.

b The service time, in minutes, is $t = 1/12$, and the number of arrivals between two successive lifts follows a Poisson with parameter $\lambda t_s = 1/3$. Therefore

$$P[N = n] = e^{-1/3}\frac{(1/3)^n}{n!}.$$

c This is the probability that two customers arrive before the first lift leaves, i.e., from above,

$$P[N = 2] = e^{-1/3}\frac{(1/3)^2}{2!}.$$

d In this case, at most one customer should arrive before the lift 1 leaves, and no one between the first and the second. I.e.,

$$(P[N = 0]+P[N = 1])P[N = 0] = [e^{-1/3}+e^{-1/3}(1/3)]e^{-1/3} = \frac{4e^{-2/3}}{3}$$

Exercise 4

a See Ross (2014, Section 6.3). Importantly:

- all times to births and deaths are independent and exponentially distributed random variables;
- when there are n persons in the system the birth and death rates are λ_n and μ_n, respectively, with $n = 0, 1, \dots$.

b $\mu_n = n\mu$;

$\lambda_n = n\lambda + \theta$.

c See Ross (2014, Example 6.4)

d The contribution of $\frac{\theta}{\lambda-\mu}[e^{(\lambda-\mu)t} - 1]$ is related to the immigration. If the immigration is 0, this term disappears.

The contribution of $ie^{(\lambda-\mu)t}$ only depends on the birth and death rates and on the initial number of elements.

If the birth rate is larger than the death rate, the size of the population continuously increases. Otherwise, without the contribution of the immigration, it shrinks.