# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

Day of examination: Monday 11. june 2018

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Formulae note for STK1100 and STK1110. Accepted calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

## Problem 1

Consider a discrete time Markov chain $\{X_n, n = 0, 1, 2, , 3, ...\}$ with state space $\{1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{bmatrix} p & q & 0 & 0 \\ 0.3 & 0.2 & r & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

parametrised by $p, q, r$. That is, $p, q, r$ are possible numbers such that $P$ is a transition probability matrix. The following questions must be answered in terms of the possible values of the parameters, which may change the basic structure of the chain as they vary. **Hint:** This Markov chain can be parametrised solely by the values of the parameter $q$.

**a** (weight 10p)

Describe the Markov chain using a diagram.

**b** (weight 10p)

Find all communicating classes of the Markov chain. Which are closed? Are there any absorbing states?

**c** (weight 10p)

Define a recurrent and a transient state. Which are the recurrent and the transient states?

**d** (weight 10p)

Calculate for all $i, j = 1, ..., 4$ the probability that $X_2 = j$ given that $X_0 = i$

*(Continued on page 2.)*

**e** (weight 10p)

Conditioned upon the chain has entered one of the states 3 or 4 find the stationary distribution over these two states.

**f** (weight 10p)

Starting in state 2 find the expected time until entering one of the recurrent states

**g** (weight 10p)

Starting in state 2 what is the probability of ultimate absorption in the states $\{3, 4\}$?

## Problem 2

**a** (weight 10p)

Define a Poisson proces $\{N(t)\}_{t\geq 0}$ with rate $\lambda > 0$ using the infinitesimal properties of its increments (if you give an alternative definition of a Poisson process, not using the infinitesimal properties of the increments, you will only get 75% of the points).

**b** (weight 10p)

Let $g(t) = \mathbb{E}\left[e^{-uN(t)}\right]$, $u \geq 0$ be the Laplace transform of $N(t)$. Prove that $g(t)$ satisfies the following differential equation

$$g'(t) = g(t)\,\lambda\left(e^{-u} - 1\right), \qquad g(0) = 1,$$

and conclude that, for $t > 0$, $N(t)$ is distributed as a Poisson random variable with parameter $\lambda$, i.e.,

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, 3, ....$$

**c** (weight 10p)

Let $S_n$ be the $n$-th arrival time for $\{N(t)\}_{t\geq 0}$. Prove that the density of $S_n$ is given by

$$f_{S_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t > 0.$$

**d** (weight 10p)

Patients arrive at the waiting room of a doctor's office according to a Poisson process with rate 6 patients an hour. The doctor starts to examine the patients only when the third patient arrives.

1. Compute the expected time from the opening of the doctor's office until the first patient starts to be examined.
2. Compute the probability that in the first opening hour the doctor does not start examining at all.

*(Continued on page 3.)*

## Problem 3

**a** (weight 10p)

Define a standard Brownian motion $\{B(t)\}_{t\geq 0}$.

**b** (weight 10p)

Compute the autocovariance function of $\{B(t)\}_{t\geq 0}$, that is, $\mathbb{E}[B_t B_s]$ for $s, t \geq 0$.

**c** (weight 10p)

Let $a \geq 0$, show that $\{X(t) = B(t+a) - B(a)\}_{t\geq 0}$ is a standard Brownian motion.

SLUTT