UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK2130 — Modelling by stochastic processes

Day of examination: Monday 11. june 2018

Examination hours: 14.30 – 18.30

This problem set consists of 7 pages.

Appendices: None

Permitted aids: Formulae note for STK1100 and STK1110.
Accepted calculator

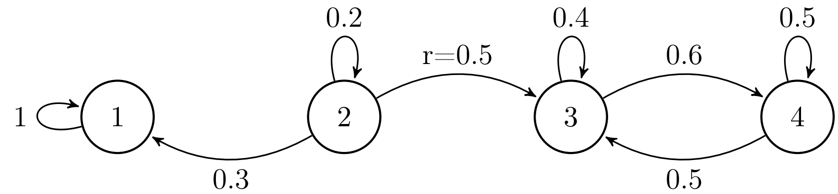
Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

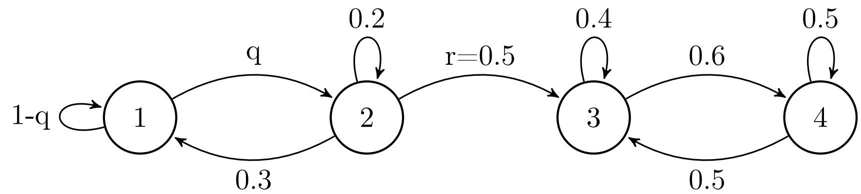
a (weight 10p)

Note that $r + 0.3 + 0.2 = 1$, which implies that $r = 0.5$. On the other hand, $p, q \in [0, 1]$ must satisfy $p + q = 1$ and, hence, we can write $p = 1 - q$ and leave $q \in [0, 1]$ as the only free parameter. Qualitatively, there are two main cases, $q = 0$ and $q \in (0, 1]$.

1. **Case** $q = 0$.



2. **Case** $q \in (0, 1]$.



b (weight 10p)

We have:

1. **Case** $q = 0$. From the diagram is clear that the communicating classes are $\{1\}, \{2\}, \{3, 4\}$. In addition $\{3, 4\}$ is closed and 1 is an absorbing state.

(Continued on page 2.)

2. **Case** $q \in (0,1]$. From the diagram is clear that the communicating classes are $\{1,2\}, \{3,4\}$. In addition $\{3,4\}$ is closed and there are no absorbing states.

c

For any state i we let f_i denote the probability that, starting in state i, the process will ever reenter state i. State i is said to be recurrent if $f_i = 1$ and transient if $f_i < 1$. Recurrence/transience is a communicating class property. Finite and closed communicating classes are recurrent. If a communicating class is not closed it must be transient.

1. **Case** $q = 0$. $\{1\}$ and $\{3,4\}$ are recurrent. $\{2\}$ is transient.
2. **Case** $q \in (0,1]$. $\{1,2\}$ is transient and $\{3,4\}$ is recurrent.

d (weight 10p)

We have that for all $i, j = 1, ..., 4$

$$P(X_2 = j | X_0 = i) = P^2_{i,j}.$$

The two-step transition probability matrix is given by

$$P^2 = PP = \begin{pmatrix} p^2 + 0.3q & 0.2q + pq & 0.5q & 0 \\ 0.06 + 0.3p & 0.04 + 0.3q & 0.3 & 0.3 \\ 0 & 0 & 0.46 & 0.54 \\ 0 & 0 & 0.45 & 0.55 \end{pmatrix}$$

e (weight 10p)

When $X_0 = 3$ or $X_0 = 4$ the process starts in a recurrent and finite class. Hence, we can regard the states $\{3,4\}$ as an irreducible finite state Markov chain with transition probability matrix

$$Q = \begin{pmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{pmatrix}.$$

As any irreducible finite state Markov chain is positive recurrent, the invariant distribution exists and is given by the solution of the following system of linear equations

$$\pi_3 = 0.4\pi_3 + 0.5\pi_4,$$
$$\pi_3 + \pi_4 = 1,$$

which is

$$\pi_3 = \frac{5}{11}, \qquad \pi_4 = \frac{6}{11}.$$

(Continued on page 3.)

f (weight 10p)

We have that the vector k^A of mean hitting times of a subset of states A is given by the minimal non-negative solution to the system

$$\begin{aligned} k_i^A &= 0, \qquad i \in A, \\ k_i^A &= 1 + \sum_{j \notin A} P_{i,j} k_j^A \qquad i \notin A \end{aligned}$$

The answer for this question is k_2^A.

1. **Case** $q = 0$. Let A=$\{1\} \cup \{3, 4\}$, then

$$k_2^A = 1 + P_{2,2} k_2^A = 1 + 0.2 k_2^A,$$

which yields $k_2^A = 10/8$.

2. **Case** $q \in (0, 1]$. Let A=$\{3, 4\}$, then

$$\begin{aligned} k_1^A &= 1 + P_{1,1} k_1^A + P_{1,2} k_2^A = 1 + (1 - q) k_1^A + q k_2^A \\ k_2^A &= 1 + P_{2,1} k_1^A + P_{2,1} k_1^A = 1 + 0.3 k_1^A + 0.2 k_2^A, \end{aligned}$$

which yields

$$\begin{aligned} k_1^A &= \frac{2\,(q + 0.8)}{q} \\ k_2^A &= \frac{2\,(q + 0.3)}{q}. \end{aligned}$$

g (weight 10p)

We have that the vector h^A of hitting probabilities of a subset of states A is given by the minimal non-negative solution to the system

$$\begin{aligned} h_i^A &= 1, \qquad i \in A, \\ h_i^A &= \sum_j P_{i,j} h_j^A \qquad i \notin A, \end{aligned}$$

The answer for this question is h_2^A with $A = \{3, 4\}$.

1. **Case** $q = 0$. Let A=$\{3, 4\}$, then

$$\begin{aligned} h_3^A &= h_4^A = 1, \\ h_1^A &= P_{1,1} h_1^A + P_{1,2} h_2^A + P_{1,3} h_3^A + P_{1,4} h_4^A = h_1^A \\ h_2^A &= P_{2,1} h_1^A + P_{2,2} h_2^A + P_{2,3} h_3^A + P_{2,4} h_4^A \\ &= 0.3 h_1^A + 0.2 h_2^A + 0.5. \end{aligned}$$

The first equation $h_1^A = h_1^A$ gives us no information. However, as the state 1 is absorbing we have that $h_1^A = 0$. Substituting this value in the last equation gives $h_2^A = 5/8$.

(Continued on page 4.)

2. **Case** $q \in (0,1]$. Let A=$\{3,4\}$, then

$$\begin{aligned} h_3^A &= h_4^A = 1, \\ h_1^A &= P_{1,1}h_1^A + P_{1,2}h_2^A + P_{1,3}h_3^A + P_{1,4}h_4^A \\ &= (1-q)\,h_1^A + qh_2^A \\ h_2^A &= P_{2,1}h_1^A + P_{2,2}h_2^A + P_{2,3}h_3^A + P_{2,4}h_4^A \\ &= 0.3h_1^A + 0.2h_2^A + 0.5, \end{aligned}$$

which yields $h_1^A = h_2^A = 1$. Note that we do not need to solve the previous equation. Actually, as $\{1,2\}$ is transient and $\{3,4\}$ is recurrent we must have $h_1^A = h_2^A = 1$.

Problem 2

A suggested solution is:

a (weight 10p)

A counting process $\{N(t)\}_{t\geq 0}$ is said to be a Poisson process with rate $\lambda > 0$ if the following axioms hold:

1. $N(0) = 0$.
2. $\{N(t)\}_{t\geq 0}$ has independent increments.
3. $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$.
4. $P(N(t+h) - N(t) \geq 2) = o(h)$.

b (weight 10p)

We start computing $g(t+h)$.

$$\begin{aligned} g(t+h) &= \mathbb{E}\left[e^{-uN(t+h)}\right] \\ &= \mathbb{E}\left[e^{-u\{N(t+h)-N(t)+N(t)\}}\right] \\ &= g(t)\,\mathbb{E}\left[e^{-u\{N(t+h)-N(t)\}}\right], \end{aligned}$$

where in the last equality we have used axiom ii. . Next we compute an $o(h)$ approximation for $\mathbb{E}\left[e^{-u\{N(t+h)-N(t)\}}\right]$. To simplify the notation let $Z = N(t+h) - N(t)$. First note that by axioms iii. and iv.

$$\begin{aligned} P(Z=0) &= 1 - \{P(Z=1) = 2 + P(Z \geq 2)\} \\ &= 1 - \{\lambda h + o(h) + o(h)\} \\ &= 1 - \lambda h + o(h). \end{aligned}$$

Therefore, conditioning on the events $\{Z=0\}, \{Z=1\}$ and $\{Z \geq 2\}$ we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-uZ}\right] &= \mathbb{E}\left[e^{-uZ}|Z=0\right]P(Z=0) + \mathbb{E}\left[e^{-uZ}|Z=1\right]P(Z=1) \\ &\quad + \mathbb{E}\left[e^{-uZ}|Z\geq 2\right]P(Z\geq 2) \\ &= 1 \times (1-\lambda h + o(h)) + e^{-u} \times (\lambda h + o(h)) + \mathbb{E}\left[e^{-uZ}|Z \geq 2\right] \times o(h) \\ &= 1 + \lambda\left(e^{-u} - 1\right)h + o(h). \end{aligned}$$

(Continued on page 5.)

Hence,

$$g(t+h) = g(t)\left\{1 + \lambda\left(e^{-u} - 1\right)h + o(h)\right\},$$

which can be written as

$$\frac{g(t+h) - g(t)}{h} = g(t)\,\lambda\left(e^{-u} - 1\right) + \frac{o(h)}{h},$$

and letting $h \to 0$ we can conclude that $g(t)$satisfies the desired differential equation. This differential equation is linear and has the solution g

$$\begin{aligned} g(t) &= g(0)\, e^{\lambda\left(e^{-u}-1\right)t} \\ &= e^{\lambda\left(e^{-u}-1\right)t}. \end{aligned}$$

On the other hand, the Laplace transform of a Poisson random variable characterizes its distribution. Hence, we only need to check that the Laplace transform of $W \sim \text{Poiss}(\lambda)$ coincides with $g(t)$. We have

$$\begin{aligned} \mathbb{E}\left[e^{-uW}\right] &= \sum_{k=0}^{\infty} e^{-uk} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\left(e^{-u}\lambda t\right)^k}{k!} \\ &= e^{-\lambda t} e^{e^{-u}\lambda t} \\ &= e^{\lambda\left(e^{-u}-1\right)t} = g(t). \end{aligned}$$

c (weight 10p)

As $\{N(t) \geq n\}$ if and only if $\{S_n \leq t\}$ one has that the distribution function of S_n is given by

$$F_{S_n}(t) = P(S_n \leq t) = P(N(t) \geq n) = \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

$$=$$

Taking derivatives we get its density

$$\begin{aligned} f_{S_n}(t) &= -\lambda \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= -\lambda \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda \sum_{k=n-1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}. \end{aligned}$$

d (weight 10p)

Let $\{N(t)\}_{t\geq 0}$ be the process counting the number of arrivals at the surgery. $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate 6 per hour. Let $\{S_n\}_{n\geq 0}$ be the the n-th arrival time, which is distributed as a $\Gamma(n, \lambda)$, i.e., the density of S_n is given by

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t > 0.$$

(Continued on page 6.)

1. The expected time is given by $\mathbb{E}[S_3] = \frac{3}{\lambda} = \frac{1}{2}$.

2. The required probability is given by

$$\begin{aligned} P(N(1) \leq 2) &= \sum_{i=0}^{2} e^{-6\times 1} \frac{(6\times 1)^i}{i!} \\ &= e^{-6}\left\{1 + \frac{6}{1} + \frac{36}{2}\right\} \\ &= e^{-6} 25 \approx 0.0620 \end{aligned}$$

Problem 3

A suggested solution is:

a (weight 10p)

A stochastic process $\{B(t)\}_{t\geq 0}$ is said to be a Brownian motion if

1. $B(0) = 0$.
2. $\{B(t)\}_{t\geq 0}$ has stationary and independent increments.
3. For every $t > 0$, $B(t)$ has normal distribution with mean zero and variance t.

b (weight 10p)

Let $s < t$, we have that

$$\begin{aligned} \mathbb{E}[B(t)B(s)] &= \mathbb{E}[B(t)B(s) + B(s)B(s) - B(s)B(s)] \\ &= \mathbb{E}[(B(t) - B(s))B(s)] + \mathbb{E}[B(s)B(s)] \\ &= \mathbb{E}[(B(t) - B(s))]\,\mathbb{E}[B(s)] + s \\ &= s, \end{aligned}$$

where in the third equality we have used that B has independent increments and $\mathbb{E}[B(s)B(s)] = \text{Var}[B(s)] = s$. Moreover, in the fourth equality we have used that $\mathbb{E}[B(s)] = 0$. We can do the same argument for $t < s$ and conclude that

$$\mathbb{E}[B(t)B(s)] = \min(s,t).$$

c (weight 10p)

One has that $X(0) = B(a) - B(a) = 0$. Let $t_1 < t_2 < \cdots < t_n$, by using that $\{B(t)\}_{t\geq 0}$ has stationary and independent increments, we can write

$$\begin{aligned} &(X(t_1), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})) \\ &= (B(a+t_1) - B(a), B(a+t_2) - B(a+t_1), ..., B(a+t_n) - B(a+t_{n-1})) \\ &\sim (B(a+t_1-a), B(a+t_2-a-t_1), ..., B(a+t_n-a-t_{n-1})) \\ &= (B(t_1), B(t_2-t_1), ..., B(t_n-t_{n-1})) \\ &\sim (B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})), \end{aligned}$$

(Continued on page 7.)

where $\sim$ means equality in law. Hence we have shown that the increments of $\{X(t) = B(t+a) - B(a)\}_{t \geq 0}$ have the same joint law as of those of a standard Brownian motion and therefore they are independent and stationary. In particular, we have shown that the law of $X(t)$ is the same as the law of $B(t) \sim N(0,t)$ and we can conclude.

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