Solution proposal STK2130-sp19

Problem 1

a) A class is defined as a set of states which communicate.

One class is $\{2,4\}$ since the paths $2,4,2,$ and $4,2,4$ are possible so 2 and 4 communicate

The second class is $\{1,5\}$ since the paths $1,5,1,$ and $5,1,5$ are possible so 1 and 5 communicate. The state $\{3\}$ is absorbing and therefore is a class of its own.

If the chain enters $\{1,5\}$, it stays there so there is an infinite number of visits to this class and the class is recurrent. If the initial state is 2 the path $2,4,5$ is possible and the chain does not return to 2. Hence the class $\{2,4\}$ is transient. The state $\{3\}$ is absorbing, and hence is recurrent since the chain stays in 1 always, i.e. an infinite number of times.

b) From the Chapman-Kolmogorov equations

$$P^2_{45} = \sum_{k=1}^{5} P_{4k}P_{k5} = (0, 1/4, 1/4, 1/4, 1/4)\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/4 \\ 1/2 \end{pmatrix} = 1/16 + 2/16 = 3/16.$$

c) The transition matrix for the chain moving outside $\{1,3,5\}$ and being absorbed in $\{1,3,5\}$ is when the states are $\{2,4,A\}$

$$Q = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P(X_3 = 2, X_k \notin \{1,3,5\}, k = 1,2|X_0 = 4) = Q^3_{42}$ But

$$\begin{aligned} Q^3 &= \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/8 & 1/8 & 3/4 \\ 1/16 & 3/16 & 6/8 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

so

$$Q^3_{42} = \sum_{k=1}^{5} Q_{4k}Q^2_{k2} = (1/4, 1/4, 1/2)\begin{pmatrix} 1/8 \\ 1/16 \\ 0 \end{pmatrix} = (1/4)(1/8) + (1/4)(1/16) = 3/64.$$

Remark that there are only two possible paths: 4,4,4,2 and 4,2,4,2 with probabilities (1/4)(1/4)(1/4)=1/64 and (1/4)(1/2)(1/4)=1/32.

d)

$$\begin{array}{rc} & P(X_3=1, X_k \notin \{1,3,5\}, k=1,2|X_0=4). \\ = & P(X_3=1, X_2=2, X_1 \notin \{1,3,5\}|X_0=4) \\ + & P(X_3=1, X_2=4, X_1 \notin \{1,3,5\}|X_0=4) \\ = & P(X_3=1|X_2=2, X_1 \notin \{1,3,5\}, X_0=4)P(X_2=2, X_1 \notin \{1,3,5\}, |X_0=4) \\ + & P(X_3=1|X_2=4, X_1 \notin \{1,3,5\}, X_0=4)P(X_2=4, X_1 \notin \{1,3,5\}|X_0=4) \\ = & = Q^2_{42}P_{21} + Q^2_{44}P_{41} \end{array}$$

where $P(X_3 = 1|X_2 = 2, X_1 \notin \{1,3,5\}, X_0 = 4) = P_{21} = 1/2$ and $P(X_3 = 1|X_4 = 2, X_1 \notin \{1,3,5\}, X_0 = 4) = P_{41} = 0$ follow by the Markov property. Thus $Q^2_{42}P_{21} + Q^2_{44}P_{41} = (1/16)(1/2) = 1/32$ since $Q^2_{42} = 1/16$

Here there is only one possible path: 4,4,2,1 with probability (1/4)(1/4)(1/2)=1/32.

e) If $X_0 \in \{1,3,5\}$, $T = 0$ so $\mu_i = E[T|X_0 = i] = 0, k \in \{1,3,5\}$. Also

$$\begin{aligned} \mu_2 &= \sum_{i=1}^{5} E[T, X_1 = i|X_0 = 2] \\ &= \sum_{i=1}^{5} E[T|X_1 = i, X_0 = 2]P(X_1 = i|X_0 = 2) \\ &= \sum_{i=1}^{5} 1 + E[T|X_0 = i]P(X_1 = i|X_0 = 2) \end{aligned}$$

since by the Markov property $E[T|X_k = i, X_{k-1} = j] = 1 + E[T|X_{k-1} = i, X_{k-2} = j] = 1 + E[T|X_{k-1} = i]$. Thus $\mu_2 = 1 + \mu_1 P_{22} + \mu_4 P_{24} = 1 + \mu_4/2$. Similarly $\mu_4 = 1 + \mu_2 P_{42} + \mu_4 P_{44} = 1 + \mu_2/4 + \mu_4/4$. The equations

$$\begin{aligned} \mu_2 &= 1 + \mu_4/2 \\ \mu_4 &= 1 + \mu_2/4 + \mu_4/4 \end{aligned}$$

have solutions $\mu_2 = 2$ and $\mu_4 = 2$.

5) The class $\{1,5\}$ is a closed class so once the chain enters the class it stays there. Hence π_5 is the limit of the proportion of time the chain is in state 5. Similarly $\pi_1 = \lim_{n\to\infty} P(X_n = 1|X_0 = 1)$ is the limit of the proportion of time the chain is in state 1. (π_1, π_5) is the solution of the equations

$$(\pi_1, \pi_5) = (\pi_1, \pi_5) \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

and $pi_1 + \pi_5) = 1$ so $(\pi_1, \pi_5) = (1/2, 1/2)$.

Problem 2

a) A birth and death process is a continuous time Markov chain with state space $0, 1, 2, \ldots$. When the chain is in state i the times until the next change of state are independent exponentially distributed with mean $1/v_i$ where $v_0 = \lambda_0$ and $v_i = \lambda_i + \mu_i$, $i = 1, 2, \ldots$. The move to the next state is described by a binary random variable which is independent of how long the chain is in state i and has a distribution where the probability that the change is to $i+1$ is $\lambda_i/(\lambda_i+\mu_i)$ he probability that the change is to $i-1$ is $\mu_i/(\lambda_i+\mu_i)$ $i = 1, 2, \ldots$ and the probability that the move is to 1 if $i = 0$ is 1.

b) The state is the number of customers so the state space is $0, 1, \ldots, s$. The chain moves from i to i+1 $i = 0, 1, \ldots, s-1$ when a new customer arrives so $\lambda_i = \lambda$ $i = 0, \ldots, s-1$. If no server is free so the state is $i = s$ the new customer leaves so $\lambda_s = 0$. If i servers are busy the chain moves from i to $i-1$, $i = 1, \ldots, s$ when the first server is free. This variable is the minimum of i independent exponentially distributed variable, which is an exponentially distributed variable with expectation $1/i\mu$

Hence $v_0 = \lambda$, $v_i = \lambda + i\mu$, $i = 1, \ldots, s-1$ and $v_s = s\mu$. The transition matrix of the jumps has elements 0 except $P_{0,1}$ and $P_{i,i+1} = \lambda/(\lambda + i\mu)$, $P_{i,i-1} = \mu/(\lambda + i\mu)$, $i = 1, \ldots, s-1$ and $P_{s,s-1} = 1$.

The instantaneous transition rates are therefore $q_{01} = \lambda$, $q_{i,i+1} = \lambda$, $i = 1, \ldots, s-1$ $q_{i,i-1} = i\mu$, $i = 1, \ldots, s-1$ and $q_{s,s-1} = s\mu$.

c) The Kolmogorov backward equations have the form

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

With the results from part b)

$$\begin{aligned} P'_{0j}(t) &= \lambda P_{1j}(t) - \lambda P_{0j}(t) \\ P'_{i,j}(t) &= \lambda P_{i,i+1}(t) + i\mu P_{i-1,j}(t) - (\lambda + i\mu) P_{i,j}(t), \ i = 1, \ldots, s-1 \\ P'_{s,j}(t) &= s\mu P_{s-1,j}(t) - s\mu P_{sj}(t). \end{aligned}$$

d) The balance equations are

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

where P_j are the limiting probabilities. In this case

$$\begin{aligned} v_0 P_0 &= q_{10} P_1 && \text{i.e. } \lambda P_0 = \mu P_1 \\ v_i P_i &= q_{i-1,i} P_{i-1} + q_{i+1,i} P_{i+1} && \text{i.e. } (\lambda + i\mu) P_i = \lambda P_{i-1} + (i+1)\mu P_{i+1}, \ i = 1, \ldots, s-1 \\ v_s P_s &= q_{s-1,s} P_{s-1} && \text{i.e. } s\mu P_s = \lambda P_{s-1}. \end{aligned}$$

e) First $P_1 = \frac{\lambda}{\mu}P_0$. Inserting in the next equation $(\lambda+\mu)P_1 = \lambda P_0 + 2\mu P_2$, i.e. $(\lambda+\mu)P_1 = \mu P_1 + 2\mu P_2$ or $P_2 = \frac{\lambda}{2\mu}P_1$. If $P_i = \frac{\lambda}{i\mu}P_{i-1}$, $(\lambda+i\mu)P_i = (i+1)\mu P_{i+1} + \lambda P_{i-1} = (i+1)\mu P_{i+1} + i\mu P_i$, so $P_{i+1} = \frac{\lambda}{(i+1)\mu}P_i$ for $i = 1,\ldots,s-1$. Also $P_s = \frac{\lambda}{s\mu}P_{s-1}$.

Hence, $P_i = \prod_{j=0}^{i} \frac{\lambda}{\mu}\cdots\frac{\lambda}{i\mu} = (\frac{\lambda}{\mu})^i \frac{1}{i!}P_0$ and $P_i = \frac{(\frac{\lambda}{\mu})^i \frac{1}{i!}}{\sum_{j=0}^{s}(\frac{\lambda}{\mu})^j \frac{1}{j!}}$.

Problem 3

a) It is impossible for the chain to return to 0 after an odd nunber of steps since it moves one step up with probability p or one step down with probability 1-p.

If the chain returns to 0 after 2n steps, there must have been n steps up and n steps down. There are $\binom{2n}{n}$ possible locations of the n steps upward and each of them occur with probability p. Hence $P_{00}^{2n} = \binom{2n}{n}p^n(1-p)^n$

b) From Stirling's approximation

$$
\begin{aligned}
(2n)! &= \frac{(2n)^{2n}\sqrt{(2\pi 2n)}}{e^{2n}} = \frac{2^{2n+1}\sqrt{\pi}n^{2n+1/2}}{e^{2n}} \\
(n!)^2 &= \frac{n^{2n}2\pi n}{e^{2n}} = \frac{2\pi n^{2n+1}}{e^{2n}}.
\end{aligned}
$$

Hence $\frac{(2n)!}{(n!)^2} = \frac{2^{2n}\sqrt{\pi}}{\pi\sqrt{n}} = \frac{4^n}{\sqrt{\pi n}}$ and$P_{00}^n = \binom{2n}{n}p^n(1-p)^n$ is approximately $\frac{[4p(1-p)]^n}{\sqrt{\pi n}}$. Therefore $\sum_{n=1}^{\infty} P_{00}^n = \infty$ if and only if $p = 1/2$ and the chain is recurrent if $p = 1/2$ and transient if $p \neq 1/2$.

c) By decomposing the event $T_1 < \infty$ according to whether $X_1 = 1$ or $X_1 = -1$ $f = P(T_1 < \infty | X_0 = 0) = P(T_1 < \infty, X_1 = 1 | X_0 = 0) + P(T_1 < \infty, X_1 = -1 | X_0 = 0)$. But

$$
\begin{aligned}
& P(T_1 < \infty, X_1 = 1 | X_0 = 0) \\
= & P(T_1 < \infty, X_1 = 1, X_0 = 0)/P(X_0 = 0) \\
= & P(T_1 < \infty | X_1 = 1 X_0 = 0)P(X_1 = 1, X_0 = 0)/P(X_0 = 0) \\
= & P(T_1 < \infty | X_1 = 1)P(X_1 = 1 | X_0 = 0)
\end{aligned}
$$

where we have used the Markov property. But $P(T_1 < \infty | X_1 = 1) = 1$ and $P(X_1 = 1 | X_0 = 0) = p$, so $P(T_1 < \infty, X_1 = 1 | X_0 = 0) = p$.

Next consider $P(T_1 < \infty, X_1 = -1 | X_0 = 0)$ which by the same calculations equals $P(T_1 < \infty | X_1 = -1)P(X_1 = -1 | X_0 = 0)$. But to reach 1 starting in -1 the chain must first reach 0, i.e one unit about the starting value which has probability $P(T_0 < \infty, | X_0 = -1)$ and is equal to $P(T_1 < \infty | X_0 = 0)$. Then having reach 0 the chain must reach 1, which also has probability $f = P(T_1 < \infty | X_0 = 0)$. Thus $P(T_1 < \infty | X_1 = -1) = f^2$ since by the Markov property the two events must be independent. Knowing that $X_n = 0$

how the chain has reach this state is independent of the future behavior, so $P(T_1 < \infty | X_1 = -1)P(X_1 = -1|X_0 = 0) = f^2 q$ and

$$f = p + qf^2.$$

The fact that $p < q$ means that the chain has a drift downward. The probability $P(T_1 < \infty|X_0 = 0)$ is the same as the probability for absorption if the state 1 has been defined as an absorbing state. That this event should have probability 1 when there is a downward drift is not reasonable. Hence $f = p/q$ is the natural choice.

d) Decomposing By decomposing the event $T_0 < \infty$ according to whether $X_1 = 1$ or $X_1 = -1$ and arguing as in part c) $P(T_0 < \infty|X_0 = 0) = P(T_0 < \infty|X_1 = 1)P(X_1 = 1|X_0 = 0) + P(T_0 < \infty|X_1 = -1)P(X_1 = -1|X_0 = 0)$ But from part c) $P(T_0 < \infty|X_1 = -1) = p/q$ so $(T_0 < \infty|X_1 = -1)P(X_1 = -1|X_0 = 0) = (p/q)q = p$.

   Write $X_n = \sum Z_i$ where $Z_i = 1$ if the ith step is up and $X_i = -1$ if the ith step is down and $Z_1, Z_2, \ldots$ are independent. $E[Z_i] = p - q = 2p - 1 < 0$. By the strong law of large numbers $\frac{1}{n}\sum_{i=1}^{n} Z_i \to 2p - 1 < 0$ with probability 1. Hence $X_n \to -\infty$ with probability 1 so $P(T_0 < \infty|X_1 = 1) = 1$ and $P(T_0 < \infty|X_1 = 1)P(X_1 = 1|X_0 = 0) = p$.

   Therefore $P(T_0 < \infty|X_0 = 0) = 2p$.

Using the results from part d) the heuristic argument for choosing the solution p/q in part c) can be made rigorous. If the solution had been 1, the probability $P(T_0 < \infty|X_0 = 0)$ would have been $1 \times p + 1 \times q = 1$. Thus there would have been an infinite number of returns to 0, which contradicts that the chain is transient when $p < 1/2$.